Mathematics Practice Book

Prerequisite mathematics knowledge for all hard science students

Contents

How to use this practice book?

- Select the chapters with the contents that you find most difficult or have never heard of. If you are a bad chooser, just start with chapter 1 or take our diagnostic mathematics test (if you have been invited to) in order to guide your chapter selection.
- For each selected chapter, we advise you to:
  1. Read through the theory of the chapter.
  2. Pay special attention to the examples: use pen, pencil and paper to follow each example, copy and complete complicated calculations, and try to think about what similar examples could look like and how to solve these.
  3. Work through the exercises without using a calculator and without looking at our solutions.
  4. Work through the exercises again and compare your own solutions with ours. Note down any remaining questions.

> These boxes contain very important (calculation) rules you should know by heart! It might be very useful to write down the contents of all of these boxes onto one or two sheets of paper (which you could then put on the inside of your toilet door to give yourself and your roommates a second important purpose during toilet visits).

- As soon as you start feeling confident about your mathematics, try to finish the tests from the final chapter with a 100% score.
- If you feel like you could use some help in getting your mathematics skills up to speed, you are always welcome to join our Warm-Up Week in the last week of August. More information about this week can be found on `https://staff.science.ru.nl/KoenvanAsseldonk/wuw.html`.

We hope that you are going to enjoy doing mathematics (or at least appreciate this practice book) and we wish you good luck with your studies!

This practice book was based on the contents of the excellent 'Arithmetic' booklet by Wim Gielen. Modifications and expansions were made by Anne-Linge Huiskamp, Batuhan Özyürek, Denise Vonck, Emma Need, Nick Rientjes, Roy van Alst and Tim Heesterbeek and coordinated by Koen van Asseldonk. If you have suggestions or if you spotted mistakes, we would be very happy to receive them on Koen.vanAsseldonk@ru.nl.

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CHAPTER 1

Numbers and basic arithmetic operations

Natural numbers. The oldest and most important set of numbers, which you have probably met very early in your education carreer, are the natural numbers. These are all positive whole numbers including zero and they can be used to 'count things'. We often abbreviate the set of all natural numbers by $\mathbb{N}$:

$$\boxed{\mathbb{N} = \{\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5\ ,\ 6\ ,\ 7\ ,\ 8\ ,\ 9\ ,\ 10\ ,\ 11\ ,\ 12\ ,\ 13\ ,\ 14\ ,\ 15\ ,\ \cdots\ \}}$$

Integers. If we include the *negative* versions of all natural numbers, we obtain the set of integers $\mathbb{Z}$:

$$\boxed{\mathbb{Z} = \{\ \cdots\ ,\ -5\ ,\ -4\ ,\ -3\ ,\ -2\ ,\ -1\ ,\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5\ ,\ 6\ ,\ 7\ ,\ 8\ ,\ \cdots\ \}}$$

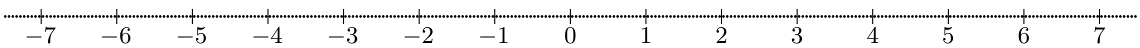
The curly brackets in the expressions for $\mathbb{N}$ and $\mathbb{Z}$ indicate that we are dealing with *sets*, whose *elements* are listed in between the curly brackets. Mathematicians often use the following notation:

Notation:	We say:	Meaning:
$x \in \mathbb{N}$	'x is an element of $\mathbb{N}$'	x is a natural number
$x \in \mathbb{Z}$	'x is an element of $\mathbb{Z}$'	x is an integer

Rational numbers. We can expand $\mathbb{Z}$ by also including fractions: numbers of the type $\dfrac{x}{y}$ where x (the numerator) and y (the denominator) are integers, and y is not zero. For example:

$$\frac{7}{4} \qquad \frac{-288}{41} \qquad \frac{126}{168}$$

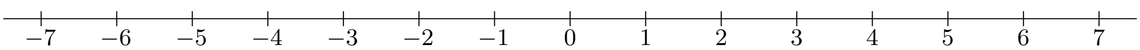
Fractions are also called rational numbers and the set of all rational numbers is denoted by $\mathbb{Q}$ (from 'quotient'). Based on the decimal notation of a number you can give every rational number a spot on a (to both sides infinitely long) number line:



Concepts like $>$ and $\leq$ tell us something about the relative positions of numbers on this line, for example:

x is positive	means	x is located to the right of 0	so	$x > 0$
x is negative	means	x is located to the left of 0	so	$x < 0$
x is smaller than y	means	x is located to the left of y	so	$x < y$
x is greater than y	means	x is located to the right of y	so	$x > y$

Real numbers. We can go a step further by looking at *all* points on the number line. The set of all these points is called the real numbers and denoted by $\mathbb{R}$:



Whereas natural numbers can be used to 'count things', real numbers can be used to 'measure things'. Most of the real numbers lack a name and fame, but due to their great significance to science, some of them managed to be named and acquired a spot on an advanced calculator:

$$\begin{aligned} e &= 2.71828182845904523536028747135266249775\cdots \\ \pi &= 3.14159265358979323846264338327950288419\cdots \end{aligned}$$

Complex numbers. We can even go another step further by expanding the real number line into a second dimension and including the imaginary numbers. All real and imaginary numbers combined are called complex numbers and their set is denoted by $\mathbb{C}$. We will not discuss complex numbers here, but you will learn all about them in your future mathematics courses.

Addition. The sum of two small natural numbers is again a natural number. Some of these sums can be found in the table (and hopefully somewhere in your head too):

Addition

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Adding larger numbers has been partly forgotten by humanity, spoiled by lightning fast calculators. It works as follows:

$$\begin{array}{r} 87654 \\ 63283 \\ + \overline{150937} \end{array}$$

Start on the right: $4+3=7$. Continue with $5+8=13$, you write down the 3 and use the 1 later. Next step: $6+2=8$, together with the 1 from your memory this gives us 9. This is how you continue.

You can use the same procedure to add more than two numbers. I can calculate $639+1257+718$ as follows:

$$\begin{array}{r} 639 \\ 1257 \\ 718 \\ + \overline{2614} \end{array}$$

Start on the right: $9+7+8=24$ (write down 4 and use 2 later). Then do $3+5+1=9$ plus the remaining 2 is 11 (write down 1, use 1 later). Continue like this.

You can also add numbers that are not natural, but the result will often be much less elegant. Rational numbers with a finite decimal continuation can be added similarly to natural numbers if you make sure to properly align the decimal points:

$$\begin{array}{r} 1.3 \\ 42.285 \\ 0.0703 \\ + \overline{43.6553} \end{array}$$

However, in most cases you will not be able to simplify the result of a sum. For example:

$$\begin{array}{rcl} 2+\pi & \text{is just} & 2+\pi \\ 4+\dfrac{1}{2}+e+\sqrt{3} & \text{is just} & 4+\dfrac{1}{2}+e+\sqrt{3} \end{array}$$

You might be tempted to represent $2+\pi$ by something along the lines of 5.14159265. This, however, is only a *rounded* version of your answer, and unless you are *asked* to round your answer, it is completely false. Please do your lecturers, TAs and yourself a favour: you are going to devote yourself to the exact natural sciences, so always present your answers in exact form (unless instructed otherwise).

Subtraction. $23 - 15 = 8$ just means $23 = 8 + 15$. You can calculate the difference of large numbers as follows:

$$\begin{array}{r} 87654 \\ -\;\underline{63283} \\ 24371 \end{array}$$

Right: $4 - 3 = 1$. Then $5 - 8 =$ panic, so I borrow a one: $15 - 8 = 7$. Then $6 - 2 = 4$, now I return the borrowed one: 3. Continue like this.

Calculation rules. The following rules should not surprise you:

$$\begin{array}{ll} x + y = y + x & \text{(addition is commutative)} \\ (x + y) + z = x + (y + z) & \text{(addition is associative)} \\ \text{if } x + y = x + z \text{ then } y = z & \\ x + 0 = x & x - (y + z) = x - y - z \\ x - x = 0 & x - (y - z) = x - y + z \end{array} \tag{1.1}$$

These rules hold for all natural numbers, integers, rational numbers and real numbers. If you are working with negative numbers, the following additional rules will help you out:

$$-(-x) = x \qquad 0 - x = -x \qquad x + (-y) = x - y \tag{1.2}$$

Multiplication. The product of two small natural numbers is again a natural number:

Multiplication

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Multiplication by a larger number is done as follows:

$$\begin{array}{r} 397 \\ \times\;\underline{6} \\ 2382 \end{array}$$

Start on the right: $6 \times 7 = 42$, you write down 2 and use 4 later. Then $6 \times 9 = 54$, plus that 4 gives 58, you write down 8 and use 5 later. Finally $6 \times 3 = 18$, plus that 5 gives 23.

If you have to calculate the product of two large numbers without a calculator:

$$\begin{array}{r} 397 \\ \times\;\underline{86} \\ 2382 \\ +\;\underline{3176} \\ 34142 \end{array}$$

First you calculate $6 \times 397 = 2382$, then $8 \times 397 = 3176$. You add these results after you shift 3176 one place to the left. This gives us the final sum to evaluate: $2382 + 31760 = 34142$

Scientists often use a dot to write down a product, so $6 \cdot 7 = 42$ just means $6 \times 7 = 42$. We can sometimes even omit the multiplication sign: $6x = 6 \cdot x$ and $ab = a \cdot b$. Please don't do this if you are multiplying two integers, because 24 is just twenty-four and this is obviously not equal to $2 \cdot 4 = 8$.

Multiplication rules. The following rules apply to all real numbers x, y and z:

(1.3)

$$\begin{array}{lll} xy = yx & 0 \cdot x = 0 & 1 \cdot x = x \\ x(y+z) = xy + xz & \text{(multiplication is distributive over addition)} & \\ x(y-z) = xy - xz & \text{(multiplication is distributive over subtraction)} & \\ xy = 0 \iff x = 0 \text{ or } y = 0 & & \\ x \cdot (-y) = -xy & -(x+y) = -x-y & -(x-y) = y-x \end{array}$$

You will probably understand these rules. How can you see for example that $3(b+c) = 3b+3c$? Well, if you had three cheese sandwiches for breakfast, then you've eaten three slices of bread and three slices of cheese.

Division. With a little bit of luck you can divide a natural number by another natural number, ending up with another natural number. If you don't know the result by heart, you can use long division for this:

$$\begin{array}{r} 3\,/\,174\,\backslash\,58 \\ -\,\underline{15} \\ 24 \\ -\,\underline{24} \\ 0 \end{array}$$

3 fits 5 times in 17 with 2 remaining.
I copy the 4. 3 fits exactly 8 times in 24.

I use the notation $174 : 3 = 58$
or $174/3 = 58$ or $\frac{174}{3} = 58$

If you are not left with a zero at the bottom, which will unfortunately happen in most of the cases, the result of your division is not a natural number:

$$\begin{array}{r} 29\,/\,743\,\backslash\,25 \\ -\,\underline{58} \\ 163 \\ -\,\underline{145} \\ 18 \end{array}$$

$743 : 29 = 25$ with remainder 18

Other notations:
$25\frac{18}{29}$ $\quad 25 + \frac{18}{29}$ $\quad$ or just $\frac{743}{29}$

Maybe you have been taught in the past to report the result of $743 : 29$ as $25\frac{18}{29}$, but it is much better to avoid these 'mixed numbers', because your TA could also understand $25\frac{18}{29}$ to be $25 \cdot \frac{18}{29}$, which is something completely different. That is why we advise you to write down $25 + \frac{18}{29}$ instead, or just leave $\frac{743}{29}$ to be your final answer.

If you paid close attention to the first page of this booklet, you might have recognised $\frac{743}{29}$ as a fraction or a rational number. That should come as no surprise, since we divided the natural number 743 by the natural number 29, and we defined fractions to be just that: two natural numbers divided by one another. We will come back to fractions in chapter 3.

Golden rule of algebra. As soon as you start studying the natural sciences, you will soon see most of the numbers being replaced with unknown variables, such as x, p, a, ... In many cases you will be asked to *solve an equation* for one of these unknown variables, which means that you are required to express this unknown variable in terms of all other numbers, constants and variables in the equation. This is where you can show your mastery of the basic arithmetic operations of addition, subtraction, multiplication and division, as long as you stick to the golden rule of algebra:

(1.4) Do unto one side of the equation what you do to the other!

For example, if you are asked to solve the equation $2x - 3 = 11$ for x, you can proceed as follows:

Add 3 to both sides of the equation: $\quad 2x - 3 + 3 = 11 + 3 \iff 2x = 14$

Divide both sides by 2: $\quad \dfrac{2x}{2} = \dfrac{14}{2} \iff \boxed{x = 7}$

Exercises chapter 1

Exercise 1. Which of the following statements are true?

a) $-2 \in \mathbb{N}$ b) $\dfrac{1}{3} \in \mathbb{R}$ c) $5 \in \mathbb{Q}$ d) $\sqrt{2} \in \mathbb{Q}$ e) $\pi \in \mathbb{C}$

Exercise 2. Calculate $283 + 1729$.

Exercise 3. Calculate $635 \cdot 728 - 208 \cdot 728$.

Exercise 4. Solve for x: $673 + 4x = 841$

Exercise 5. Solve for α: $\dfrac{2\alpha - 3}{\alpha + 1} = 3$

Exercise 6. Solve for p: $\dfrac{2}{p} + 1 = \dfrac{3}{2p} + 1$

CHAPTER 2

Powers

Squares, cubes, ... If we multiply a real number x repeatedly by itself, we can nicely abbreviate our multiplication using a power of x:

x^2	(we say: x squared)	means	$x \cdot x$	for example $7^2 = 49$
x^3	(we say: x cubed)	means	$x \cdot x \cdot x$	for example $7^3 = 343$
x^4	(we say: x to the fourth)	means	$x \cdot x \cdot x \cdot x$	for example $7^4 = 2401$
x^5	(we say: x to the fifth)	means	$x \cdot x \cdot x \cdot x \cdot x$	for example $7^5 = 16807$

This can be continued for higher powers of x (x to the sixth, x to the seventh, et cetera). The raised number on the right of the x is called the exponent of x.

Calculation rules. For all real numbers x and y and for all natural exponents n and m, you can use:

$$x^n \cdot x^m = x^{n+m} \qquad (xy)^n = x^n y^n$$
$$\frac{x^n}{x^m} = x^{n-m} \quad \text{(only if } x \neq 0\text{)} \qquad (x^n)^m = x^{nm} \tag{2.1}$$

I hope that you not only memorize these rules, but also understand them. How do we derive for example $k^2 \cdot k^3 = k^5$? Well, a row of two kittens followed by a row of three kittens results in a row of five kittens:

$$k^2 \cdot k^3 = (k \cdot k) \cdot (k \cdot k \cdot k) = k \cdot k \cdot k \cdot k \cdot k = k^5$$

And why is $\left(k^2\right)^3 = k^6$? Because a row of three kitten couples is nothing else than a row of six kittens. So if you want to calculate 7^8 without a calculator, you can either go the hard way (multiply 7 by 7, then multiply the result by 7, then multiply the result by 7, ...) or you can smartly use the calculation rule $x^{nm} = (x^n)^m$ twice:

$$7^8 = (7^4)^2 = ((7^2)^2)^2 = (49^2)^2 = 2401^2 = 5764801$$

Power of 0. You might wonder what happens if $m = n$ in the calculation rule $\frac{x^n}{x^m} = x^{n-m}$. Let's see what happens to the left-hand side:

$$\frac{x^n}{x^m} = \frac{x^n}{x^n} = 1$$

And what happens to the right-hand side?

$$x^{n-m} = x^{n-n} = x^0$$

So now we know what x to the power of 0 is:

$$x^0 = 1 \tag{2.2}$$

This rule is valid for all real values of x except 0, because nobody knows what 0^0 is.

Expanding brackets. You will very often encounter powers of algebraic expressions with brackets containing unknown variables, for example $(2x-1)^2$ or $(3xy)^3$. It is sometimes convenient to expand these kind of expressions by repeated application of the calculation rules from this chapter and the previous chapter:

$$(2x-1)^2 = (2x-1)(2x-1) = 2x(2x-1) - 1(2x-1) = 4x^2 - 2x - 2x + 1 = 4x^2 - 4x + 1$$
$$(3xy)^3 = 3^3 \cdot x^3 \cdot y^3 = 27x^3y^3$$

Notable products. The following expansions are so incredibly important that you should learn them by heart and never ever forget them:

$$\begin{aligned} (x+y)^2 &= x^2 + 2xy + y^2 \\ (x-y)^2 &= x^2 - 2xy + y^2 \\ (x+y)(x-y) &= x^2 - y^2 \end{aligned} \tag{2.3}$$

Exercises chapter 2

Exercise 1. Express y in terms of x: $y(x+2) = (x+3)(y-1)$

Exercise 2. Prove the notable products and learn them by heart.

Exercise 3.

a) Which of the following identities are not correct?

$x^3+x^4 = x^7$ $\qquad$ $(x+y+z)^2 = x^2+y^2+z^2$ $\qquad$ $x-(y-z) = x-y-z$ $\qquad$ $(x+y)^3 = x^3+y^3$

b) Simplify the left-hand sides of the incorrect identities in a) as much as possible.

Exercise 4. Calculate $5^3 + 5^4$.

Exercise 5. Simplify the expression $(a+3)^2 - (a-3)^2$.

Exercise 6. Calculate $\dfrac{9^{27}}{3^{50}}$.

Exercise 7. Simplify the expression $\dfrac{a^2-b^2}{a+b}$.

Exercise 8. Calculate $\dfrac{378^2-1}{377}$.

Exercise 9. Determine all integers x that satisfy $x^3 = 25x$.

Exercise 10. Which is greater, 2^{30} or 3^{20} ?

Exercise 11. Write $(x+1)^3 - (x-1)^3$ as simple as possible.

CHAPTER 3

Fractions

In chapter 1 we have seen that fractions are numbers of the type $\frac{x}{y}$ where x (the numerator) and y (the denominator) are integers, and y is not zero. Although $\frac{x}{y}$ is the most convenient notation for fractions, we sometimes use different ways to express them:

$7:4$	instead of	$\frac{7}{4}$
$82/25$	instead of	$\frac{82}{25}$
$3\frac{5}{7}$	instead of	$3+\frac{5}{7}$ (but, as explained on page 5, please don't use the notation $3\frac{5}{7}$)
0.3	instead of	$\frac{3}{10}$
3.28	instead of	$3+\frac{2}{10}+\frac{8}{100}$

All calculation rules from the previous chapters are valid when solving problems with fractions, more so:

$$\frac{x}{1}=x \qquad \frac{ax}{ay}=\frac{x}{y} \qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \qquad \frac{x}{a}+\frac{y}{a}=\frac{x+y}{a} \tag{3.1}$$

It is important to keep in mind that these rules only work if the denominator is not zero! If you happen to run into a denominator that *is* zero, you are not dealing with a fraction anymore (check the definition in chapter 1), which means that you can't use these calculation rules.

Simplification of fractions. Occasionally, you can use $\frac{ax}{ay}=\frac{x}{y}$ to simplify a fraction. For example:

$$\frac{126}{168}=\frac{2\cdot 63}{2\cdot 84}=\frac{63}{84}=\frac{3\cdot 21}{3\cdot 28}=\frac{21}{28}=\frac{7\cdot 3}{7\cdot 4}=\frac{3}{4}$$

It is often desired to simplify your fractions as much as possible, which means that you make the numerator and the denominator as small as possible. That essentially means that you have to divide both the numerator and the denominator by their *greatest common divisor* (gcd), which is the greatest number that both parts of the fraction can be divided by to yield integers in the numerator and the denominator.

Addition of fractions. I have not given any rule for simplifying $\dfrac{a}{b}+\dfrac{c}{d}$. So how do you add two fractions nevertheless?

1. Rewrite the fractions so they have equal denominators using the rule $\dfrac{x}{y}=\dfrac{ax}{ay}$.
2. Then apply the rule $\dfrac{p}{r}+\dfrac{q}{r}=\dfrac{p+q}{r}$.

For example, if you want to add $\frac{7}{12}$ and $\frac{13}{42}$, follow these steps:

$$\frac{7}{12}+\frac{13}{42}=\frac{7\cdot 7}{12\cdot 7}+\frac{13\cdot 2}{42\cdot 2}=\frac{49}{84}+\frac{26}{84}=\frac{49+26}{84}=\frac{75}{84}=\frac{3\cdot 25}{3\cdot 28}=\frac{25}{28}$$

If you think this approach is difficult, you will have to derive a calculation rule for the sum of two fraction and resort to memorising this terrible thing (see exercise 4).

Quotient of fractions. It is also possible to divide one fraction by another. For example:

$$\frac{\frac{16}{3}}{\frac{2}{5}}=\frac{5\cdot 3\cdot\frac{16}{3}}{5\cdot 3\cdot\frac{2}{5}}=\frac{5\cdot 16}{3\cdot 2}=\frac{80}{6}=\frac{40}{3}$$

You might find it clever to define a rule based on this method:

$$\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \qquad \text{(dividing by a fraction is equal to multiplication by the inverse)} \tag{3.2}$$

The calculation above will now become

$$\frac{\frac{16}{3}}{\frac{2}{5}}=\frac{16}{3}\cdot\frac{5}{2}=\frac{16\cdot 5}{3\cdot 2}=\frac{80}{6}=\frac{40}{3}$$

Cross-multiplication. The golden rule of algebra allows us to derive the following rule:

$$\frac{a}{b} = \frac{c}{d} \quad \Longleftrightarrow \quad ad = bc$$

We call this trick 'cross-multiplication'. You can not only use it to test whether two fractions are equal, but you can also use it while solving equations with fractions.

Powers of fractions. If we combine the calculation rules from the previous chapter with those from this chapter, we can derive a useful additional calculation rule:

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{3.3}$$

In addition, we can also allow exponents to be negative integers using the following definition:

$$x^{-n} \quad \text{means} \quad \frac{1}{x^n} \tag{3.4}$$

Some examples:

$$\left(\frac{2}{3}\right)^4 = \frac{16}{81} \qquad 5^{-3} = \frac{1}{125} \qquad \left(\frac{3}{7}\right)^{-1} = \frac{7}{3}$$

Of course, it's still impossible to divide by zero, so these calculations rules work for all real numbers x and y and all integers n as long as you are not secretly trying to divide by zero.

Decimal notation. Some fractions have a finite decimal notation (for example $\frac{82}{25} = 3.28$), but usually the decimal continuation is infinite. For example, let's try to calculate $\frac{3604}{495}$ using long division:

$$\begin{array}{l}
495 \,/\, 3604 \qquad \backslash\, 7.280808\cdots\cdots \\
\quad -\underline{3465} \\
\qquad 1390 \\
\quad\; -\underline{990} \\
\qquad\; 4000 \\
\qquad -\underline{3960} \\
\qquad\quad 4000 \\
\qquad\; -\underline{3960} \\
\qquad\qquad 4000 \\
\qquad\quad -\underline{3960} \\
\qquad\qquad\;\; \text{etc}
\end{array}$$

You discover $\frac{3604}{495} = 7.28080808080808080808080808080808080808080808080808080808\cdots$ and you will probably understand why the decimal continuation of a rational number always ends with such a periodic tail: when doing a long division only a finite amount of rests are possible. Calculators usually try to fool you with something along the lines of $\frac{3604}{495} = 7.280808081$ while hoping for you to forgive them for this minor imperfection.

Fractions with variables. All rules and procedures for fractions with numbers that we just discussed are equally applicable to fractions with variables. For example:

$$\frac{6x^2+3x}{3x} = \frac{3x(2x+1)}{3x} = 2x+1 \quad \text{if} \quad x \neq 0 \qquad \text{(simplification)}$$

$$\frac{1+x^2}{2x-2} + \frac{3-x^2}{x^2-x} = \frac{x\cdot(1+x^2)}{x\cdot(2x-2)} + \frac{2\cdot(3-x^2)}{2\cdot(x^2-x)} = \frac{x+x^3}{2x^2-2x} + \frac{6-2x^2}{2x^2-2x} = \frac{x+x^3+6-2x^2}{2x^2-2x} \qquad \text{(addition)}$$

$$\frac{\frac{2-3x}{3-x}}{\frac{1+2x}{3-x}} = \frac{2-3x}{3-x}\cdot\frac{3-x}{1+2x} = \frac{2-3x}{1+2x} \quad \text{if} \quad x \neq 3 \qquad \text{(division)}$$

Don't forget that no fraction can ever have a denominator of zero, so if you simplify fractions with variables, always check if there exist values of your variables for which the simplifying step is invalid. If so, you should of course report those values.

Long division with variables. It can be hard to simplify fractions with powers of x in both the numerator and the denominator. If the degree (which is the highest power of x) of the numerator is greater than or equal to the degree of the denominator, you can use long division to simplify the fraction. For example:

$$\frac{3x^2+2x-1}{x-1} = 3x+5+\frac{4}{x-1}$$

$$\begin{array}{llllllll}
x-1 & / & 3x^2 & + & 2x & - & 1 & \backslash \quad 3x+5+\frac{4}{x-1} \\
 & - & 3x^2 & - & 3x & & & \\
\hline
 & & & & 5x & - & 1 & \\
 & & & - & 5x & - & 5 & \\
\hline
 & & & & & & 4 &
\end{array}$$

How many times does $x-1$ fit into the highest power of x in $3x^2+2x-1$? That's $3x$ times in this case. Subtract and again check how many times $x-1$ fits into the highest power of x in what remains: 5 times. The remainder 4 cannot be divided by $x-1$ anymore.

Partial fraction decomposition. If the degree of the numerator is smaller than the degree of the denominator, you can sometimes simplify a fraction using partial fraction decomposition. This trick works if you can factor the denominator. Let's have a look at an example:

$$\frac{x+7}{x^2+2x-3}$$

The denominator x^2+2x-3 can be factored into $(x+3)(x-1)$, and the trick is to assume that we can write the original fraction as the sum of two simpler fractions, with the factors we just found as denominators:

$$\frac{\alpha}{x+3} + \frac{\beta}{x-1}$$

What are the correct values of α and β? Let's add these two simpler fractions in order to find this out:

$$\frac{\alpha}{x+3} + \frac{\beta}{x-1} = \frac{\alpha(x-1)}{(x+3)(x-1)} + \frac{\beta(x+3)}{(x-1)(x+3)} = \frac{\alpha(x-1)+\beta(x+3)}{x^2+2x-3} = \frac{(\alpha+\beta)x-\alpha+3\beta}{x^2+2x-3}$$

Now, for this fraction to be equal to our original fraction, their numerators should be equal too:

$$(\alpha+\beta)x-\alpha+3\beta = x+7$$

This equation is only true for all values of x if the coefficients of x on both sides are equal and the constants on both sides are equal:

$$\alpha+\beta = 1 \qquad \text{and} \qquad -\alpha+3\beta = 7$$

I use the first equation to eliminate β from the second:

$$\beta = 1-\alpha \quad \Longrightarrow \quad -\alpha+3\beta = -\alpha+3(1-\alpha) = -4\alpha+3 = 7 \quad \Longrightarrow \quad -4\alpha = 4 \quad \Longrightarrow \quad \alpha = -1$$

so $\beta = 1-\alpha = 1-(-1) = 2$ and we have found the partial fraction decomposition of our original fraction:

$$\frac{x+7}{x^2+2x-3} = \frac{-1}{x+3} + \frac{2}{x-1} = \frac{2}{x-1} - \frac{1}{x+3}$$

Exercises chapter 3

Exercise 1. Determine the decimal continuation of $\frac{1}{7}$.

Exercise 2. Calculate:

$$\text{a)}\ \frac{34}{22}+\frac{12}{22} \qquad \text{b)}\ \frac{5}{3}-\frac{5}{7} \qquad \text{c)}\ \frac{41}{75}-\frac{9}{20}$$

Exercise 3. Simplify the fraction $\frac{117}{819}$.

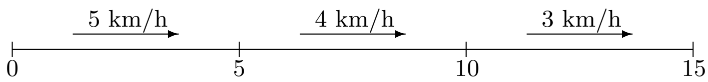
Exercise 4. Determine a calculation rule for $\dfrac{a}{b}+\dfrac{c}{d}$.

Exercise 5. Solve x from $\frac{7}{25}x \cdot \frac{3}{14} = \frac{9}{100}$.

Exercise 6. Solve x from $\dfrac{x+3}{x+7} = \dfrac{x-1}{x+1}$.

Exercise 7. If $x = \frac{7}{2}$ and $y = \frac{14}{5}$, how much is $\dfrac{x+y}{x-y}$?

Exercise 8. I'm going for a 15 kilometre walk. Initially my walking velocity is 5 km/hour, but after 5 km I'm getting tired and slow down to 4 km/hour. In the final part of the route, after the 10 kilometre mark, I crawl with only 3 km/hour to the finishing line. Calculate my average velocity.



Exercise 9. Express α in terms of β:

$$\beta = \frac{3\alpha+2}{5\alpha-1}$$

Exercise 10. The conversion from Fahrenheit to degrees Celsius goes as follows

$$C = \frac{5}{9}\,(F-32)$$

Now determine the conversion formula from Celsius to Fahrenheit.

Exercise 11. Are the following three numbers all different?

$$(-0.5)^{-2} \qquad\qquad -0.5^{-2} \qquad\qquad 0.5^2$$

Exercise 12. Simplify the expression $\dfrac{x^{-3}}{(3x^2)^{-2}}$.

Exercise 13. The number of euros x I have at time t is given by $x = \dfrac{1}{1-t}+\dfrac{1}{1+t}+\dfrac{2}{t^2-1}$. Can you find a simpler expression for how much money I have at time t?

Exercise 14. Solve the following equations for x:

$$\text{a)}\ \frac{3x+2}{3}\cdot\frac{24}{3} = 9 \qquad \text{b)}\ 0 = \frac{4}{x-2}\cdot\frac{x-6}{8} \qquad \text{c)}\ \left(\frac{5}{72}+\frac{1}{18}\right)x = \frac{200}{34}+\frac{57}{51}$$

Exercise 15. Simplify the following fractions:

$$\text{a)}\ \frac{\frac{3-3x}{4x-2}}{\frac{x-1}{1-2x}} \qquad \text{b)}\ \frac{5}{x^2-3x-4} \qquad \text{c)}\ \frac{8x^2-2x+1}{2x+3} \qquad \text{d)}\ \frac{1-x}{x^2+3x+2} \qquad \text{e)}\ \frac{2x^2-x-6}{x-2}$$

CHAPTER 4

Roots

Square roots. $\sqrt{7}$ (we say: the square root of 7) is defined as the positive real number for which the square returns 7. It is not trivial to find decimal continuations of roots. In the case of $\sqrt{7}$ you will probably be able to see that

$$\sqrt{7} = 2.6\cdots \qquad \text{because 7 lies between } 2.6^2 = 6.76 \text{ and } 2.7^2 = 7.29$$

There are more tricks available to quickly discover more decimals, but I won't bother you with those:

$$\sqrt{7} = 2.645751311\cdots$$

Notice that this is not the only number of which the square is 7, there is one more: $-2.645751311\cdots$
In general: if x is a non-negative real number, then $\sqrt{x}$ is the number that satisfies

$$\left(\sqrt{x}\right)^2 = x \qquad \sqrt{x} \geq 0 \tag{4.1}$$

The following calculation rules apply:

$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} \qquad \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}} \qquad \sqrt{x^2} = x \qquad \sqrt{x^n} = \left(\sqrt{x}\right)^n \tag{4.2}$$

Very important note: these rules only work if x and y are positive. The square root of a negative number does not exist, because squares can never be negative. The third calculation rule can be generalised to all real values of x using absolute values, which you will learn more about below.

Manipulation of square roots. Several methods exist to simplify expressions with square roots:

1. You can take squares outside of roots:

   $\sqrt{45} = 3\sqrt{5}$ because $\sqrt{45} = \sqrt{9\cdot 5} = \sqrt{9}\cdot\sqrt{5} = 3\sqrt{5}$

2. You can sometimes simplify sums or differences of roots:

   $\sqrt{18} - \sqrt{8} = \sqrt{2}$ because $\sqrt{18} - \sqrt{8} = 3\sqrt{2} - 2\sqrt{2} = \sqrt{2}$

3. You can eliminate denominators from roots:

   $\sqrt{\frac{3}{5}} = \frac{1}{5}\sqrt{15}$ because $\sqrt{\frac{3}{5}} = \sqrt{\frac{3\cdot 5}{5\cdot 5}} = \sqrt{\frac{15}{25}} = \frac{\sqrt{15}}{\sqrt{25}} = \frac{\sqrt{15}}{5} = \frac{1}{5}\sqrt{15}$

4. You can also eliminate roots from denominators:

   $\frac{3}{\sqrt{2}} = \frac{3}{2}\sqrt{2}$ because $\frac{3}{\sqrt{2}} = \frac{3\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{3\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}$

5. You can eliminate sums or differences involving roots from denominators:

   $\frac{10}{\sqrt{7}-\sqrt{2}} = 2\sqrt{7} + 2\sqrt{2}$ because $\frac{10}{\sqrt{7}-\sqrt{2}} = \frac{10\cdot\left(\sqrt{7}+\sqrt{2}\right)}{\left(\sqrt{7}-\sqrt{2}\right)\cdot\left(\sqrt{7}+\sqrt{2}\right)} = \frac{10\sqrt{7}+10\sqrt{2}}{5}$

   This so-called 'square root trick' is based on the notable product $(a-b)(a+b) = a^2 - b^2$.
   The idea is that you multiply a denominator of the form $a - b$ with $a + b$ (or vice versa)
   and in order to keep the fraction the same you should of course do the same to the numerator.
   If a or b (or both) is a root, this root will disappear in the final form of your fraction.

You should always try to simplify a root as much as possible using these methods.

Absolute value. We define the absolute value of a number as its distance to 0:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{4.3}$$

For example: $|7| = 7$ and $|-7| = 7$. The following calculation rules will probably not surprise you:

$$|xy| = |x| \cdot |y| \qquad |x^n| = |x|^n \qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad \sqrt{x^2} = |x| \tag{4.4}$$

We can now define the distance between two numbers on the real numbers line as $|x - y|$.

Cube root. $\sqrt[3]{x}$ (we say: the cube root of x) is the real number y for which $y^3 = x$. Examples:

$$\sqrt[3]{125} = 5 \qquad \sqrt[3]{-8} = -2$$

The following calculation rules are almost completely similar to those for square roots:

$$\sqrt[3]{xy} = \sqrt[3]{x} \cdot \sqrt[3]{y} \qquad \sqrt[3]{\frac{x}{y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{y}} \qquad \sqrt[3]{x^3} = x \qquad \left(\sqrt[3]{x}\right)^3 = x$$

The only difference is that we have no absolute value signs in the rule $\sqrt[3]{x^3} = x$ because cubes and cube roots can also be negative, whereas squares and square roots can never be negative.

In a similar fashion we can define $\sqrt[n]{x}$ also for larger n:

$$\begin{aligned}
\sqrt[4]{x} &= \text{the number } y \geq 0 \text{ for which } y^4 = x \quad (\text{only defined if } x \geq 0) \\
\sqrt[5]{x} &= \text{the number } y \text{ for which } y^5 = x \\
\sqrt[6]{x} &= \text{the number } y \geq 0 \text{ for which } y^6 = x \quad (\text{only defined if } x \geq 0) \\
\sqrt[7]{x} &= \text{the number } y \text{ for which } y^7 = x \\
&\vdots
\end{aligned}$$

Examples:

$$\sqrt[5]{32} = 2 \qquad \sqrt[6]{8} = \sqrt{2} \qquad \sqrt[23]{-1} = -1$$

The 'normal' $\sqrt{x}$ you may also write as $\sqrt[2]{x}$ now. The calculation rules for $\sqrt[n]{x}$ are again similar to those for square roots and cube roots, and they depend on whether n is even or odd:

(4.5)

- If n is even, x^n and $\sqrt[n]{x}$ can *never* be negative, so the calculation rules for square roots apply.
- If n is odd, x^n and $\sqrt[n]{x}$ *can* be negative, so the calculation rules for cube roots apply.

Other exponents. Roots can be translated into powers if we allow exponents to be non-integer real numbers. If we take a positive real number a and raise it to the power of x, with x being any real number, we get an expression of the form a^x with the following properties:

$$a^0 = 1 \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad a^{-x} = \frac{1}{a^x} \tag{4.6}$$

All the previously encountered calculation rules for normal exponents are once again valid for these new a^x. Examples:

$$2^{\frac{1}{3}} = \sqrt[3]{2} \qquad 4^{-\pi} = \frac{1}{4^\pi} \qquad 32^{-\frac{4}{5}} = \frac{1}{32^{\frac{4}{5}}} = \frac{1}{(32^{\frac{1}{5}})^4} = \frac{1}{(\sqrt[5]{32})^4} = \frac{1}{2^4} = \frac{1}{16}$$

Exercises chapter 4

Exercise 1. For which real numbers x is the calculation rule $\sqrt{x^2} = x$ valid?

Exercise 2. Determine the greater of the following two numbers:

$$x = \frac{\sqrt{3}}{\sqrt{5}} \qquad\qquad y = \frac{\sqrt{8}}{\sqrt{13}}$$

Exercise 3. Solve for x:

$$\sqrt{x+3} = 2\sqrt{x-1}$$

Exercise 4. Solve for x:

$$\sqrt{x+15} + \sqrt{x} = 15$$

Exercise 5. Simplify the following numbers:

a) $\sqrt{12} - \sqrt{3}$ b) $\dfrac{13}{\sqrt{5}} - \sqrt{20}$ c) $\dfrac{3+5\sqrt{2}}{1+\sqrt{2}}$ d) $\left(2+\sqrt{3}\right)^3$

Exercise 6. Determine all real numbers x that satisfy $|x+321| = |x-750|$.

Exercise 7. Write the following expressions, in which x represents a positive real number, as simple as possible:

a) $\dfrac{9-x}{3+\sqrt{x}}$ b) $\dfrac{x\sqrt{6}}{\sqrt{3x}}$ c) $\dfrac{\sqrt{3x}-\sqrt{x}}{\sqrt{3x}+\sqrt{x}}$ d) $\dfrac{x-\sqrt{x}}{1-\sqrt{x}}$

Exercise 8. Solve x from $3x = |x-1|$.

Exercise 9. Find a real number x that satisfies $\sqrt{1+x^2} = 7+x$, or prove that this is impossible.

Exercise 10. Calculate:

a) $25^{-\frac{1}{2}}$ b) $8^{\frac{2}{3}}$ c) $2^{\frac{7}{2}}/4$

Exercise 11. Solve x from:

a) $2^{x-2} = \left(\dfrac{1}{2}\right)^{2x}$ b) $8^{4x} = 32$ c) $\left(\dfrac{1}{5}\right)^{2x+5} = 125^x$

Exercise 12. Find a natural number n for which $\sqrt[9]{7^n} = \sqrt[n]{7^4}$.

Exercise 13. Silly Sally tries to prove that -7 equals 7:

$$-7 = (-7)^1 = (-7)^{2\cdot\frac{1}{2}} = \left((-7)^2\right)^{\frac{1}{2}} = 49^{\frac{1}{2}} = \sqrt{49} = 7$$

In which step did Sally try to fool you?

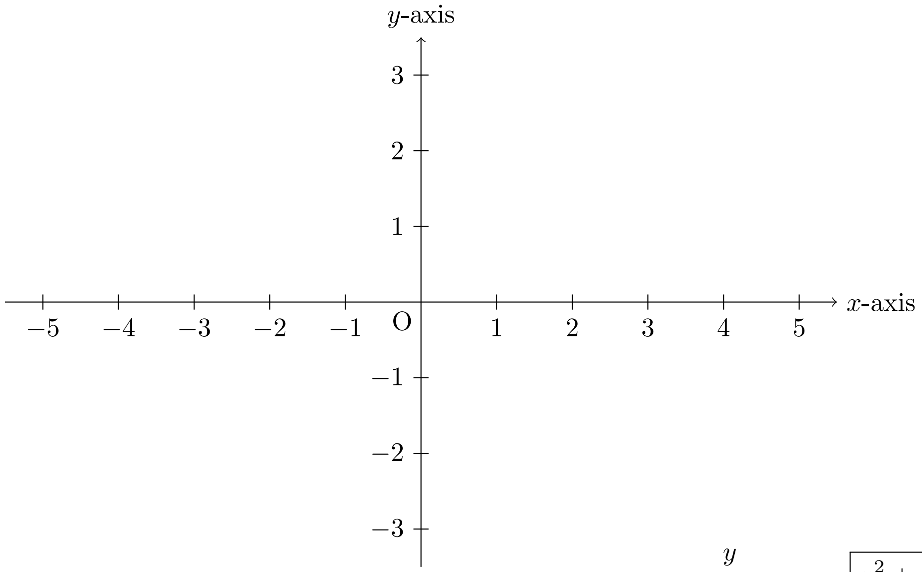
Exercise 14. We can use square roots to solve equations containing x^2. Try to solve the following equations using techniques you might have learned in high school. If you have no clue how to do this, don't worry; just return to this problem after having studied Chapter 8 of this book.

a) $3x^2 = 4$ b) $x^2 - x = 20$ c) $x^2 - 2x - 2 = 0$ d) $2x^2 + 5x = 3$

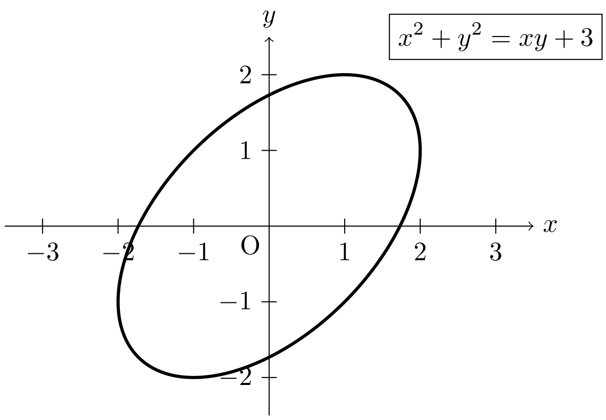
CHAPTER 5

Curves, functions and graphs

In this chapter we will be working in $\mathbb{R}^2$, which is the collection of all pairs (x, y) of two real numbers. You can interpret such a pair as a point in the plane with the following coordinate system:



Curves. The figure to the right shows all points (x, y) satisfying the equation $x^2 + y^2 = xy + 3$. Such a collection of points is what we call a 'curve' in the plane. You will encounter all sorts of curves during your studies. The curve to the right belongs to the (somewhat rare) category 'oblique ellipses'.

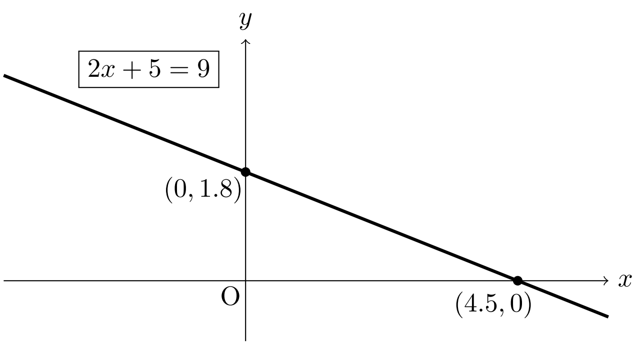


Lines. A line in $\mathbb{R}^2$ is a curve of the type

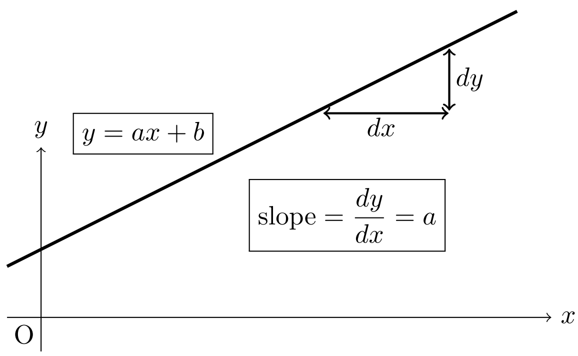
$$\alpha x + \beta y = \gamma.$$

The line to the left is the line $2x + 5y = 9$. Two remarks:

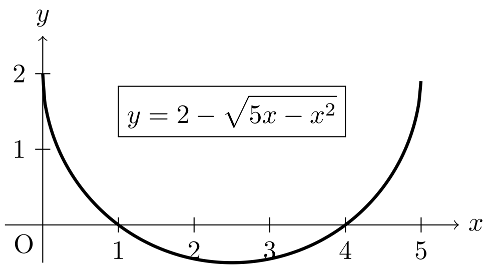
1. If $\gamma = 0$ then the line goes through the origin $(0, 0)$.
2. If $\beta \neq 0$ then you can divide the equation by β and write it in the form $y = ax + b$.



Slope. Consider a line with equation $y = ax + b$. The number a is called the <u>slope</u> of the line. The slope describes both the direction and the steepness of the line. It is the ratio between an increase dx in x and the corresponding increase dy in y.



Functions and graphs. The curve in the figure below is the collection of all points (x, y) satisfying the equation $y = 2 - \sqrt{5x - x^2}$ from $x = 0$ to and including $x = 5$:

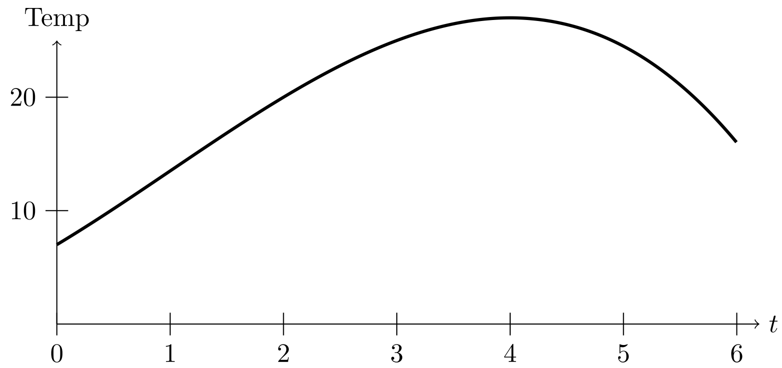


This curve has a special property: each value of x from $x = 0$ to and including $x = 5$ has exactly one single corresponding value of y on the curve. We say that y is a <u>function</u> of x on the domain $0 \le x \le 5$. The curve is then called the <u>graph</u> of this function, and the equation $y = 2 - \sqrt{5x - x^2}$ is called the <u>rule of correspondence</u> or <u>function rule</u> of this function.

Functions in practice. In daily life variables are usually not called x and y. For example,

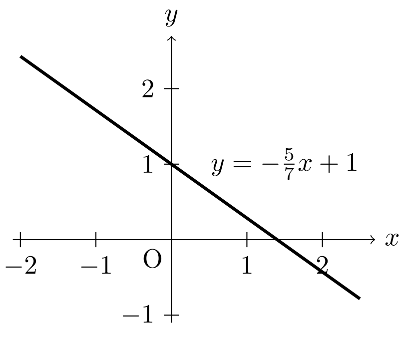
$$\text{Temp} = 7 + 6t + \frac{3}{4}t^2 - \frac{1}{4}t^3$$

could be a function rule which for every time t between 0.00 and 6.00 hours returns the temperature in degrees Celsius:

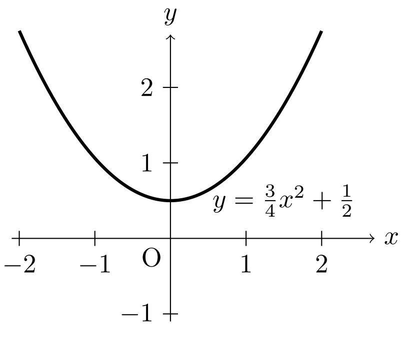


Instead of Temp, one usually writes Temp$\,(t)$ to show that the temperature depends on t. And when I write that Temp$\,(2) = 20$ I inform you that after 2 hours, the temperature was exactly 20 degrees.

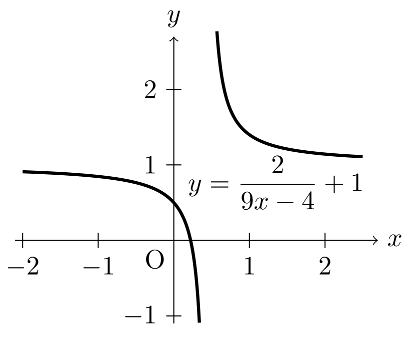
Domain and range. The <u>domain</u> of a function is the set of all input values for which the function is defined (all values x you can put into the function). The <u>range</u> of a function is the set of all values that the function takes (all values y that can come out of the function). The domain and range of a function can be determined from the function rule, but this can sometimes be rather complicated. A sketch of the graph of the function can give you a good estimate:



Domain: all real numbers ($\mathbb{R}$)
Range: all real numbers ($\mathbb{R}$)



Domain: all real numbers ($\mathbb{R}$)
Range: all real numbers $\ge \frac{1}{2}$



Domain: all real numbers except $\frac{4}{9}$
Range: all real numbers except 1

Sketching a graph. When asked to sketch a graph of a function, the goal is to get a quick impression of its general shape and scale. Not all points of the graph have to be aligned perfectly, but all the necessary information has to be available to the reader. Axes should always have a label with their name (and unit if it is a physical quantity). Similarly, the function should be depicted in the graph, either by name or function rule. Finally, all interesting points with corresponding coordinates should be indicated, such as intersections with axes and other functions, minima and maxima.

Inverse. A function f can be considered 'something' (for example a little machine) that takes numbers from its domain as its input, does some mathematical manipulations to these numbers, and returns the resulting numbers as its output. The inverse of f is another function that does the exact opposite: it takes the output from f, does some mathematical manipulations, and returns the corresponding input of f. For example:

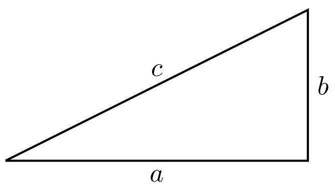
$$\text{The inverse of } f(x) = \sqrt{x} \text{ is the function } g(x) = x^2.$$
$$\text{The inverse of } f(x) = 2x+1 \text{ is the function } g(x) = \frac{x-1}{2}.$$

If the inverse of a function $f(x)$ exists, you can find it in most cases by taking its function rule, replacing $f(x)$ with y, expressing x in terms of y and formulating the resulting expression as a new function. Let's for example find the inverse of the following function:

$$f(x) = \frac{1+x}{2-x} \quad\Longrightarrow\quad y = \frac{1+x}{2-x} \quad\Longrightarrow\quad y(2-x) = 1+x \quad\Longrightarrow\quad 2y-1 = (1+y)x \quad\Longrightarrow\quad x = \frac{2y-1}{1+y}$$

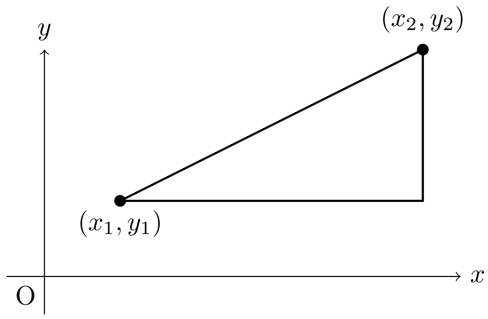
Conclusion: the inverse of the function $f(x) = \dfrac{1+x}{2-x}$ is the function $g(x) = \dfrac{2x-1}{1+x}$.

Distance. The distance of (x_1, y_1) to (x_2, y_2) equals $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.
This follows from the Pythagorean Theorem:



$a^2 + b^2 = c^2$

$\Longrightarrow$



$(x_2 - x_1)^2 + (y_2 - y_1)^2 = \text{distance}^2$

Circles. The circle with centre (p, q) and radius r consists of all points (x, y) that lie a distance r from (p, q). Its equation is therefore

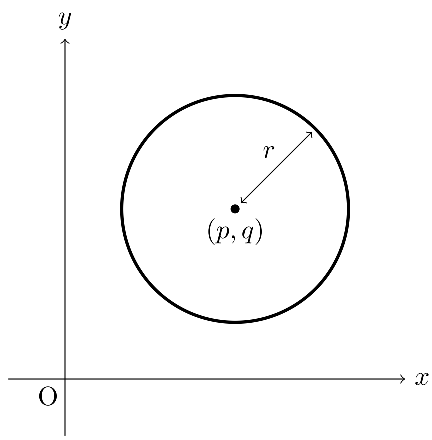
(5.1) $$(x-p)^2 + (y-q)^2 = r^2$$

You can calculate its circumference using the following formula:

(5.2) $$\text{circumference} = 2\pi r$$

Its area is given by another well-known formula:

(5.3) $$\text{area} = \pi r^2$$

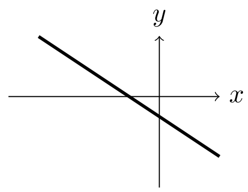


Exercises chapter 5

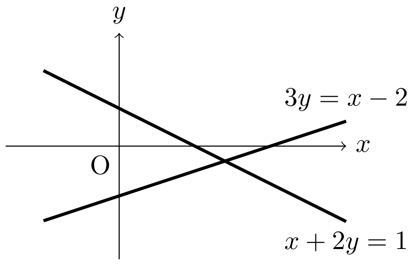
Exercise 1. Draw the curve $x^2 = 3y^2$.

Exercise 2. Determine the equation of the line through the points $(-1, 0)$ and $(2, 1)$.

Exercise 3. The line in the figure below is given by the equation $2x + 3y + 1 = 0$. What is its slope?

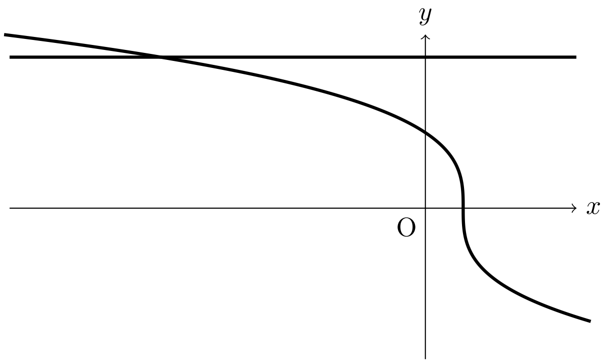


Exercise 4. Determine the intersection point of the lines $x + 2y = 1$ and $3y = x - 2$.



Exercise 5. Sketch a part of the graph of the function $y = 1 + \sqrt{x-3}$. What are the domain and the range of this function?

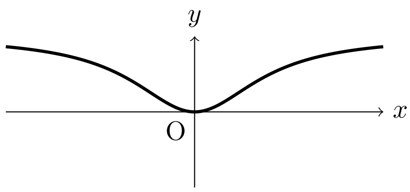
Exercise 6. Determine the intersection point of the graph $y = \sqrt[3]{1-2x}$ with the line $y = 2$.



Exercise 7. Sketch the graphs of $y = x^2$ and $y = |x|$ and find their intersection points.

Exercise 8. Sketch the graph of $y = 2^{x-3}$ and find the number x for which $y = 64$.

Exercise 9. This curve with equation $y = x^2 - yx^2$ is the graph of a function. Determine the function rule for this function.

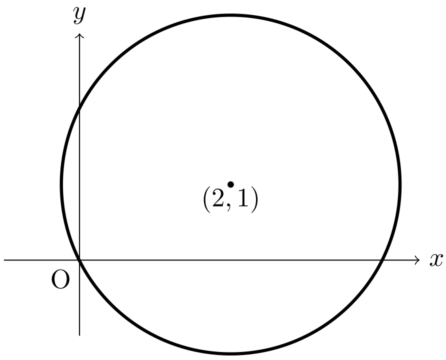


Exercise 10. Calculate the inverses of the following functions:

a) $f(x) = \dfrac{3-x}{4}$ b) $g(x) = x^2 + 2x + 1$ for $x \geq 0$ c) $h(x) = \sqrt[3]{2-x}$

Exercise 11. Calculate the distance from $(-6, 3)$ to $(5, 1)$.

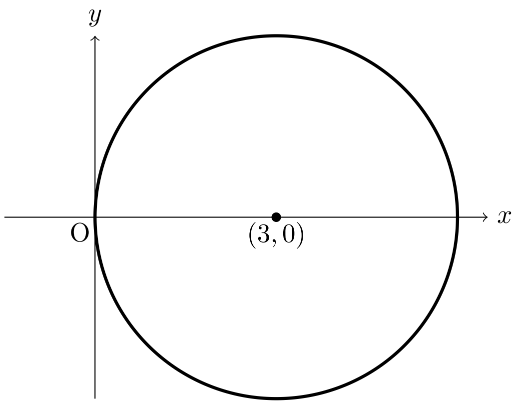
Exercise 12. Calculate the area of the circle with centre $(2, 1)$ that goes through the origin.



Exercise 13. The circle below has circumference 6π and centre $(3, 0)$

a) What is its equation?

b) Does $(5, 2)$ lie inside or outside the circle?



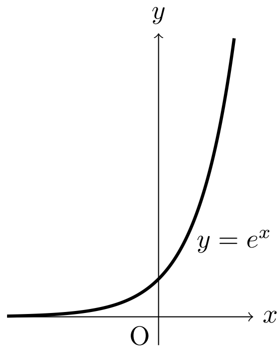
CHAPTER 6

The functions exp, ln and log

Exp. At the end of chapter 4 we have seen that you can raise a positive real number a to the power of any real number x. If we choose a to be the number $e = 2.71828182845904523536028747135266249775 7\cdots$ and we let x run over the real numbers line, we obtain the exponential function $y = e^x$. This function has an important special property: for all values of x the slope of the graph of $y = e^x$ is equal to the value of e^x. We can summarise this as follows:

$$\frac{d}{dx}\, e^x = e^x \quad \text{for all values of } x$$

You will learn what $\frac{d}{dx}$ means in chapter 9, but for now you can read it as 'the slope of' or 'the derivative of'.



This special property of $y = e^x$ is exactly the reason why the number e has received a dedicated symbol and why this function is so important in the natural sciences. There exist many quantities in nature and in your daily life of which the increase per unit time is equal to (or proportional to) the current number. You can for example think of a population of reproducing bunnies, the amount of a radioactive substance, or the number of euros on your savings account. These are all phenomena that can be described by exponential functions.

Calculation rules for exp. The following calculation rules for e^x should already be familiar to you:

(6.1) $$e^0 = 1 \qquad e^{x+y} = e^x \cdot e^y \qquad e^{x-y} = \frac{e^x}{e^y} \qquad (e^x)^y = e^{xy}$$

It is sometimes convenient to write $\exp(x)$ instead of e^x.

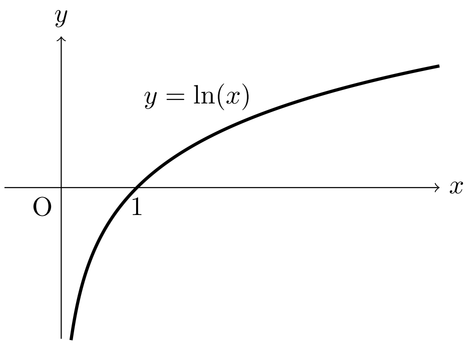
Logarithm. The 'natural logarithm' (notation: ln) is the inverse of the function $y = e^x$:

(6.2) $$y = e^x \quad \Longleftrightarrow \quad x = \ln y$$

In other words, these functions neutralise each other:

(6.3) $$\ln(e^x) = x \qquad e^{\ln x} = x$$

The domain of the function ln is the collection of all positive real numbers, and the graph of ln is the mirror image of the graph of e^x about the line $y = x$ (which makes sense, because you have to switch the roles of x and y):



Calculation rules for ln:

(6.4) $$\ln(1) = 0 \qquad \ln(xy) = \ln(x) + \ln(y) \qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y) \qquad \ln(x^y) = y\ln(x)$$

These rules follow directly from the calculation rules for e^x, but they are definitely worth memorising on their own. If you're up for the challenge, you can prove the rules yourself in exercise 14.

Tricks with ln. Two examples of how you can do calculations with ln:

Exercise 1. Solve x from $3^x = 5 \cdot 2^x$.

Solution. The golden rule of algebra says that I am allowed to do the same manipulations on both the left-hand side and the right-hand side of this equation. Let's see what happens if I take the natural logarithm of bose sides:

$$\begin{aligned} 3^x = 5 \cdot 2^x \quad &\Longrightarrow \quad \ln(3^x) = \ln(5 \cdot 2^x) \quad \Longrightarrow \quad x \ln 3 = \ln 5 + \ln 2^x \quad \Longrightarrow \quad x \ln 3 = \ln 5 + x \ln 2 \\ &\Longrightarrow \quad x \ln 3 - x \ln 2 = \ln 5 \quad \Longrightarrow \quad x(\ln 3 - \ln 2) = \ln 5 \quad \Longrightarrow \quad x = \frac{\ln 5}{\ln 3 - \ln 2} \end{aligned}$$

Exercise 2. Calculate $3^{\ln 2} - 2^{\ln 3}$.

Solution. I can use the calculation rule $x = e^{\ln x}$:

$$\left.\begin{aligned} 3^{\ln 2} &= e^{\ln\left(3^{\ln 2}\right)} = e^{\ln 2 \cdot \ln 3} \\ 2^{\ln 3} &= e^{\ln\left(2^{\ln 3}\right)} = e^{\ln 3 \cdot \ln 2} \end{aligned}\right\} \quad \Longrightarrow \quad 3^{\ln 2} - 2^{\ln 3} = 0$$

Expressing powers in exp. You can write any power in exponential form using the following rule:

$$p^q = e^{q \ln p} \tag{6.6}$$

The proof of this rule is straightforward: $p^q = e^{\ln p^q} = e^{q \ln p}$.

Logarithm with a different base. ${}^a\log x$ (defined with base $a > 0$ and $x > 0$) is the real number y for which $a^y = x$. So:

$$y = {}^a\log x \quad \Longleftrightarrow \quad a^y = x \tag{6.7}$$

You may encounter the logarithm with base 10 in old school books and on calculators (on which its name is just 'log'). The logarithm with base 2 remains popular among computer scientists. Examples:

$${}^2\log 8 = 3 \qquad\qquad {}^{10}\log 100 = 2$$

There exist calculations rules for ${}^a\log x$ similar to the ones for $\ln x$. You could also use the relationship between ${}^a\log$ and ln:

$${}^a\log x = \frac{\ln x}{\ln a} \tag{6.8}$$

Exercises chapter 6

Exercise 1. Write the following numbers as simple as possible:

a) $-\ln\dfrac{1}{7}$ b) $\ln 6 - \ln 3$ c) $\dfrac{\ln 9}{\ln 3}$ d) $\ln 2 + \ln 0.5$

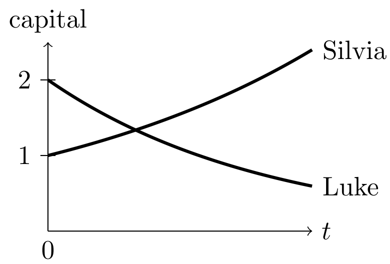
Exercise 2. Calculate ${}^{10}\log 0.001$.

Exercise 3. Draw the graph of $y = e^{-x}$ and calculate the intersection point of this curve with the line $y = 0.2$.

Exercise 4. Determine the real number x satisfying $5^{x+1} = 7^{x-1}$.

Exercise 5. (units: year, 1000 euros)
Silvia becomes richer, while Luke is getting poorer. Their capital at time t is given by:

Silvia:	$\sqrt{\exp(t)}$
Luke:	2^{1-t}



When will Silvia be just as rich as Luke is?

Exercise 6. Calculate $\left(\sqrt{e}\right)^{\ln 9}$.

Exercise 7. Simplify the expression $e^{3\ln t}$.

Exercise 8. Calculate $\ln(2e^2 + e^2)$.

Exercise 9. Calculate 49^t if $t = \dfrac{1}{\ln 7}$.

Exercise 10. Express ${}^{2}\log x$ in terms of $\ln x$.

Exercise 11. Solve x from $\ln 2x = 1 + \ln x^2$.

Exercise 12. Solve x from $e^{2x} = 20 + e^x$.

Exercise 13. After t days there are 3^t fleas. When will there be 1000 fleas?

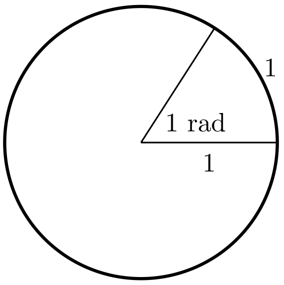
Exercise 14. Use the calculation rules for exp in (6.1) and the definition of ln in (6.2) and (6.3) to prove each of the four calculation rules for ln in (6.4).

CHAPTER 7

The functions sin, cos and tan

Degrees and radians. Angles can be measured in degrees (for example: a right angle is 90°) or in radians (for example: a right angle is $\frac{\pi}{2}$ radians). The definition of the term 'radian':

$$\boxed{\text{A radian is the size of the angle that is associated with arc length 1 at a distance 1.}}$$

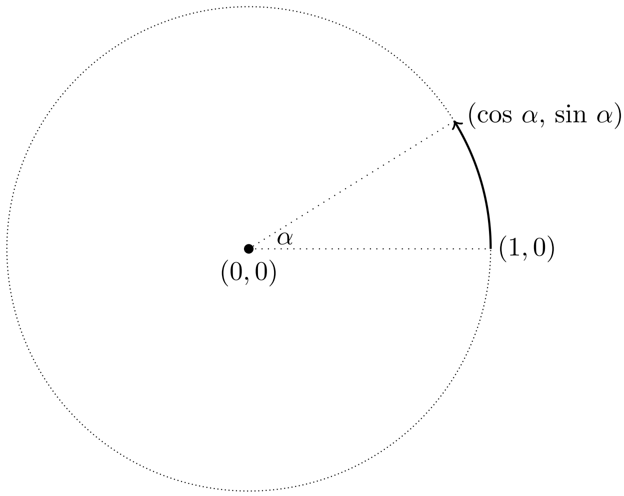


Explanation. Draw a circle with radius 1. From the centre at an angle of 1 radian, you see an arc of a circle with length 1. The circumference of the whole circle is 2π, so 360° is 2π radians and therefore:

(7.1)
$$\begin{aligned} 1 \text{ degree} &= \frac{\pi}{180} \text{ radians} \\ 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \end{aligned}$$

Sine and cosine. Suppose an ant is crawling along the circle with equation $x^2 + y^2 = 1$. This circle is called the **unit circle** because it has radius 1 and it is centred at the origin. If the ant starts crawling in $(1, 0)$ and stops after crawling over an arc length α in counterclockwise direction, it will end up in the point which creates an angle of α radians with the positive x-axis. The coordinates of this point have been given special names:

x-coordinate: **cosine** of α
y-coordinate: **sine** of α



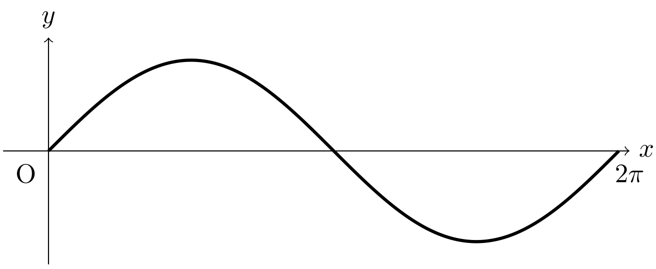
which we can write as

$$\boxed{x = \cos(\alpha) \qquad\qquad y = \sin(\alpha)}$$

For example, after crawling over an angle of $\frac{\pi}{2}$, the ant will be in the point $(0, 1)$, so $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$. This also works for negative α: the ant will then crawl clockwise. For example: $\cos(-5\pi) = -1$ and $\sin(-5\pi) = 0$, because if the crawls an arc length 5π backwards, the animal ends up in $(-1, 0)$.

If we substitute the coordinates of the point $(\cos\alpha, \sin\alpha)$ into the equation of the unit circle $x^2 + y^2 = 1$, we obtain the most important trigonometric identity: $\cos^2\alpha + \sin^2\alpha = 1$. Let's replace the arbitrary symbol α with x and reverse the order of cos and sin, so that we can put it into a box in the most common form:

(7.2)
$$\sin^2 x + \cos^2 x = 1 \qquad \text{(the Pythagorean identity)}$$

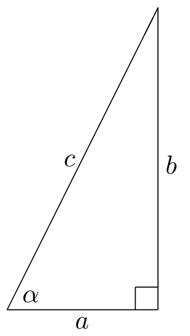


This is the graph of the sine. I only drew the part from 0 to 2π. This pattern will be continued periodically: $\sin(x + 2\pi) = \sin x$ because the crawling ant will end up in exactly the same place if it crawls for another 2π.

Tangent. $\tan x$ is the quotient of $\sin x$ and $\cos x$:

(7.3)
$$\tan x = \frac{\sin x}{\cos x}$$

Interpretation in a right-angled triangle. The following expressions hold for $0 < \alpha < \frac{\pi}{2}$:

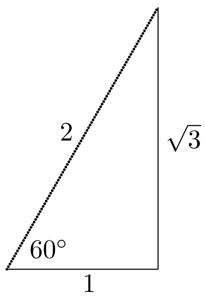


$$\cos\alpha = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{a}{c}$$

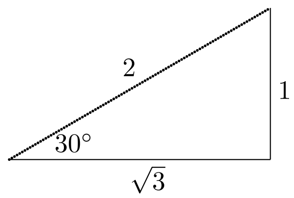
$$\sin\alpha = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan\alpha = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{b}{a}$$

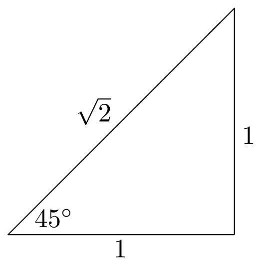
Examples.



$60°$ is $\frac{\pi}{3}$ radians, so $\boxed{\begin{aligned} \cos\frac{\pi}{3} &= \frac{1}{2} \\ \sin\frac{\pi}{3} &= \frac{1}{2}\sqrt{3} \\ \tan\frac{\pi}{3} &= \sqrt{3} \end{aligned}}$



$30°$ is $\frac{\pi}{6}$ radians, so $\boxed{\begin{aligned} \cos\frac{\pi}{6} &= \frac{1}{2}\sqrt{3} \\ \sin\frac{\pi}{6} &= \frac{1}{2} \\ \tan\frac{\pi}{6} &= \frac{1}{3}\sqrt{3} \end{aligned}}$



$45°$ is $\frac{\pi}{4}$ radians, so $\boxed{\begin{aligned} \cos\frac{\pi}{4} &= \frac{1}{2}\sqrt{2} \\ \sin\frac{\pi}{4} &= \frac{1}{2}\sqrt{2} \\ \tan\frac{\pi}{4} &= 1 \end{aligned}}$

Peculiar notation. $\sin^2 x$ means $(\sin x)^2$, which is completely different from $\sin(x^2)$, and for example $\tan^3 x$ means $(\tan x)^3$ and not $\tan(x^3)$.

Calculation rules. In addition to the Pythagorean identity and the definition of the tangent, you should know the following important calculation rules by heart: (x can be any real number and n is an integer)

$$\begin{array}{ll} \sin(n\pi) = 0 & \cos(n\pi) = (-1)^n \\ \sin(x + 2n\pi) = \sin x & \cos(x + 2n\pi) = \cos x \\ \sin(-x) = -\sin x & \cos(-x) = \cos x \\ \sin(\pi - x) = \sin x & \cos(\pi - x) = -\cos x \\ \sin x = \cos(\frac{\pi}{2} - x) & \cos x = \sin(\frac{\pi}{2} - x) \\ \sin 2x = 2\sin x\cos x & \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{array} \tag{7.4}$$

There exist many more calculation rules for sin and cos, but most of them are slightly less important, so in most cases you don't need to know them by heart (but it won't hurt if you do). For example the angle sum and difference identities:

$$\boxed{\begin{array}{ll} \sin(x + y) = \sin x\cos y + \cos x\sin y & \sin(x - y) = \sin x\cos y - \cos x\sin y \\ \cos(x + y) = \cos x\cos y - \sin x\sin y & \cos(x - y) = \cos x\cos y + \sin x\sin y \end{array}} \tag{7.5}$$

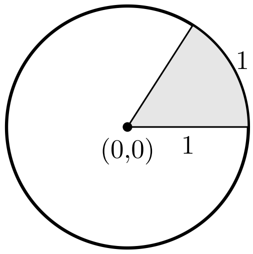
Exercises chapter 7

Exercise 1. Draw the graph of $y = \cos x$ on the domain $0 \leq x \leq 2\pi$.

Exercise 2. Fill in the missing numbers:

a) 30 degrees $= \cdots$ radians b) $\frac{5\pi}{4}$ radians $= \cdots$ degrees c) 63 degrees $= \cdots$ radians

Exercise 3. Calculate the surface area of a sector of the unit circle with an arc length of 1.



Exercise 4.

a) Calculate the exact value of the sine of 150 degrees.

b) Calculate the exact value of the cosine of $-\frac{37}{4}\pi$ radians.

Exercise 5. Calculate:

a) $\sin\dfrac{2\pi}{3}$ b) $\tan\left(-\dfrac{3\pi}{4}\right)$ c) $\tan\dfrac{\pi}{2}$ d) $\sin\dfrac{7\pi}{6}$

Exercise 6. Fill in the exact values of the missing numbers:

$$\text{If } \cos x = 0.9 \text{ then } \cos 2x = \cdots$$

$$\text{If } \sin x = 0.3 \text{ then } \cos 2x = \cdots$$

Exercise 7.

a) Solve for x on the interval $[0, 2\pi]$:

$$\sin 2x = \sin\left(x + \frac{\pi}{2}\right)$$

b) Solve for x on the interval $[-\pi, \pi]$:

$$\sin 3x = \cos 2x$$

Exercise 8. Simplify the following expressions:

a) $(\sin t + \cos t)^2 - \sin 2t$ b) $\cos^4 t - \sin^4 t$ c) $\cos 7t \cos 2t - \sin 7t \sin 2t$

Exercise 9. Express in terms of $\sin x$ and $\cos x$:

a) $\sin(2\pi - x)$ b) $\cos\left(\dfrac{3\pi}{2} - x\right)$ c) $\sin\left(\dfrac{3\pi}{4} + x\right)$ d) $\tan x + \dfrac{1}{\tan x}$

Exercise 10. Find an expression for $\cos x$ if $3\cos 2x + 5\cos x = 3$.

Exercise 11. Find the exact value of $\sin\dfrac{\pi}{12}$.

Exercise 12. Simplify the following expressions:

a) $\sin 3t + 4\sin^3 t$ b) $\dfrac{\sin 2t - \tan t}{\tan t}$

Exercise 13. Prove the useful identity

$$\boxed{1 + \tan^2 x = \frac{1}{\cos^2 x}}$$

CHAPTER 8

Equations, inequalities and systems of equations

Linear equations. The golden rule of algebra says that an equation of the type $\cdots x + \cdots = \cdots x + \cdots$ doesn't essentially change (which means that its solutions remain the same) under the following manipulations: multiplying (or dividing) both sides by a number $\neq 0$ and transfering one term to the other side of the equals sign and change the sign of this term from plus to minus (or vice versa). For example, if I want to solve for x from $\frac{3}{5}x + \frac{7}{2} = 1 - \frac{1}{4}x$, I can proceed as follows:

1. I'm not a big fan of fractions, so I multiply everything by 20:

$$\tfrac{3}{5}x + \tfrac{7}{2} = 1 - \tfrac{1}{4}x \quad \Longleftrightarrow \quad 12x + 70 = 20 - 5x$$

2. I transfer $-5x$ to the left (becomes $+5x$) and 70 to the right (becomes -70):

$$\Longleftrightarrow \quad 12x + 5x = 20 - 70 \quad \Longleftrightarrow \quad 17x = -50 \quad \Longleftrightarrow \quad x = -\tfrac{50}{17}$$

Inequalities. The same holds for inequalities (such as $-3x + 5 \leq 1 - x$), but multiplication by a negative number reverses the inequality sign, so $\leq$ becomes $\geq$ for example:

$$-3x + 5 \leq 1 - x \quad \overset{\text{sort}}{\Longleftrightarrow} \quad -3x + x \leq 1 - 5 \quad \Longleftrightarrow \quad -2x \leq -4 \quad \overset{\cdot(-\frac{1}{2})}{\Longleftrightarrow} \quad x \geq 2$$

Equations in context. If you have to solve a contextual problem, you will often have to define yourself what x means, and then construct an equation for x:

Problem. How much beer (5% alcohol) do I need to add to half a litre of wine (12% alc) to get a mixed drink with 10% alcohol?

Solution. Suppose I add x litres of beer. The alcohol percentage then becomes

$$\frac{\text{amount of alcohol}}{\text{amount of mixed drink}} \cdot 100\% = \frac{0.05\,x + 0.12 \cdot 0.5}{x + 0.5} \cdot 100\% = \frac{5x + 6}{x + 0.5} \text{ percent}$$

and the equation to solve is

$$\frac{5x + 6}{x + 0.5} = 10 \quad \Longrightarrow \quad 5x + 6 = 10(x + 0.5) = 10x + 5 \quad \Longrightarrow \quad -5x = -1 \quad \Longrightarrow \quad x = \frac{1}{5}$$

So 0.2 litres of beer need to be added to the wine.

Quadratic equations. You will probably be familiar with the quadratic formula to solve these:

$$\boxed{ax^2 + bx + c = 0 \quad \Longleftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{8.1}$$

If the discriminant $b^2 - 4ac$ is negative, the equation doesn't have any solutions. An example:

$$3x^2 - 5x - 1 = 0 \quad \Longleftrightarrow \quad x = \frac{5 \pm \sqrt{25 + 12}}{6} = \frac{5}{6} \pm \frac{1}{6}\sqrt{37}$$

So the solutions are $\frac{5}{6} + \frac{1}{6}\sqrt{37}$ and $\frac{5}{6} - \frac{1}{6}\sqrt{37}$.

Factorisation. Occasionally you can write $ax^2 + bx + c$ as a product of two factors, for example:

$$3x^2 + 5x - 2 = 0 \quad \Longleftrightarrow \quad (3x - 1)(x + 2) = 0 \quad \Longleftrightarrow \quad 3x - 1 = 0 \text{ or } x + 2 = 0 \quad \Longleftrightarrow \quad x = \tfrac{1}{3} \text{ or } x = -2$$

Completing the square. This is another useful technique to solve second degree polynomial equations. Let's look at the procedure using the example $3x^2 - 12x + 3 = 0$:

1. Make sure that the coefficient of x^2 in the equation is equal to 1:

$$3x^2 - 12x + 3 = 0 \quad \Longleftrightarrow \quad x^2 - 4x + 1 = 0$$

2. Rewrite the x^2 and x terms in the form $(x + \cdots)^2$ where the dots should be replaced by half of the coefficient of x. In this case we get $(x - 2)^2$.

3. Expand $(x+\cdots)^2$ to retrieve the x^2 and x terms from the original equation plus an additional constant:
$$(x-2)^2 = x^2 - 4x + 4$$
4. Subtract this additional constant from $(x+\cdots)^2$ and substitute the result into the original equation:
$$(x-2)^2 - 4 = x^2 - 4x \quad \xRightarrow{\text{substitute into } x^2-4x+1=0} \quad (x-2)^2 - 4 + 1 = 0$$
5. Solve the equation using standard algebraic manipulations:
$$(x-2)^2 - 4 + 1 = 0 \quad \Longleftrightarrow \quad (x-2)^2 = 3 \quad \Longleftrightarrow \quad x-2 = \pm\sqrt{3} \quad \Longleftrightarrow \quad \boxed{x = 2 \pm \sqrt{3}}$$

Another example:
$$x^2 - 10x + 18 = 0 \quad \Longleftrightarrow \quad (x-5)^2 - 7 = 0 \quad \Longleftrightarrow \quad (x-5)^2 = 7 \quad \Longleftrightarrow \quad x-5 = \pm\sqrt{7} \quad \Longleftrightarrow \quad x = 5 \pm \sqrt{7}$$

Hidden quadratic equations. The variable is in this case not x but something else, for example:
$$4^x + 2^x = 1 \quad \Longleftrightarrow \quad (2^x)^2 + 2^x - 1 = 0 \quad \Longleftrightarrow \quad \left(2^x + \frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0 \quad \Longleftrightarrow \quad 2^x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$
We can forget about the negative solution, because 2^x is never negative, so
$$2^x = \frac{-1+\sqrt{5}}{2} \quad \Longrightarrow \quad \ln 2^x = \ln \frac{-1+\sqrt{5}}{2} \quad \Longrightarrow \quad x \ln 2 = \ln \frac{-1+\sqrt{5}}{2} \quad \Longrightarrow \quad x = \frac{\ln \frac{-1+\sqrt{5}}{2}}{\ln 2}$$

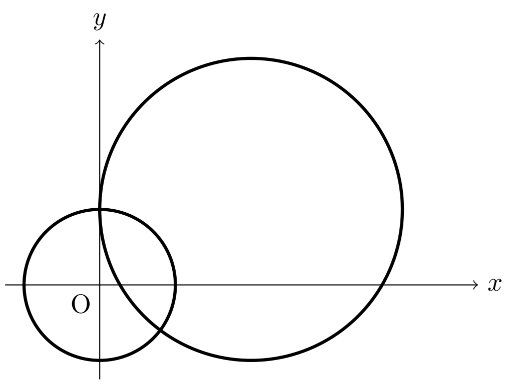
System of equations. We consider two strategies for two or more equations with two or more unknowns:

1. Eliminate one of the unknowns.
2. Express one of them in terms of the other and substitute.

I'll give you an example of both strategies:

Example 1. Solve for x and y from
$$\boxed{\begin{aligned} 2x + 7y &= 3 \\ 3x + 5y &= 2 \end{aligned}}$$

Solution. I'm going to eliminate x, which I manage to do if I multiply the first equation by 3 and the second by 2, and then subtract the second equation from the first:
$$\left.\begin{aligned} 2x + 7y = 3 \quad &\xRightarrow{\cdot 3} \quad 6x + 21y = 9 \\ 3x + 5y = 2 \quad &\xRightarrow{\cdot 2} \quad 6x + 10y = 4 \end{aligned}\right\} \quad \xRightarrow{-} \quad 11y = 5 \quad \Longrightarrow \quad y = \tfrac{5}{11}$$
Once you know y you can use either of the two equations to determine x: $x = -\frac{1}{11}$.



Example 2. Determine the points where the circles $x^2 + y^2 = 1$ and $(x-2)^2 + (y-1)^2 = 4$ intersect.

Solution. Expanding the brackets in the equation of the second circle yields
$$x^2 - 4x + 4 + y^2 - 2y + 1 = 4$$
which can be rearranged to
$$x^2 + y^2 = 4x + 2y - 1$$
The intersection points must be on both circles, so I can substitute $x^2 + y^2 = 1$, yielding
$$4x + 2y - 1 = 1 \quad \Longrightarrow \quad y = 1 - 2x$$
If I substitute this in $x^2 + y^2 = 1$, I get
$$x^2 + (1-2x)^2 = 1 \quad \Longrightarrow \quad 5x^2 - 4x = 0 \quad \Longrightarrow \quad x(5x-4) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = \frac{4}{5}$$
So the intersection points are $(0, 1)$ and $(\frac{4}{5}, -\frac{3}{5})$.

Exercises chapter 8

Exercise 1. Solve the following equations:

a) $\frac{3}{2}x + \frac{5}{7} = 1 - x$ b) $-2(x-5) = 3(2-3x) + 5(1-x)$ c) $\frac{1-4x}{x-3} = 5$

Exercise 2. How much wine (12% alc) do I have to add to one litre of beer (5% alc) to obtain a drink with 8% alcohol?

Exercise 3. Evaluate which real numbers x satisfy $1 < \frac{7}{2} - \frac{1}{6}x < 2$

Exercise 4. Solve:

a) $x^2 = 1 + 2x$ b) $x^2 = 6x + 7$ c) $3x^2 + 5x = (x+1)^2$

Exercise 5. Solve the equation $x^2 + 3x + 1 = 0$ by completing the square.

Exercise 6. Calculate the maximum value of $5x - x^2$ by completing the square.

Exercise 7. Find the number t between 0 and π for which

$$2\sin t = \sqrt{5 - 4\cos t}$$

Exercise 8. Determine all real numbers x that satisfy the equation

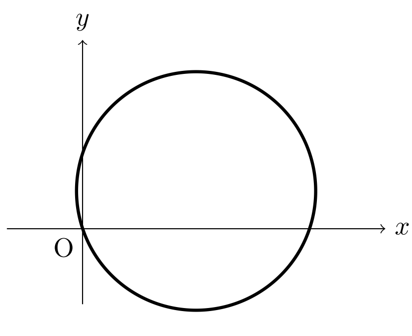
$$x = 2\sqrt{x} + 3$$

Exercise 9. (units: hours, metres)
A ladybug and a beetle are crawling along the x-axis between $t = 0$ and $t = 1$ with their position for every t given by

$$\boxed{\begin{array}{lcl} \text{ladybug}\,(t) & = & 6 - e^t \\ \text{beetle}\,(t) & = & e^{2t} \end{array}}$$

At what time will they meet?

Exercise 10. The circle that is pictured to the right satisfies the equation $x^2 + y^2 = 3x + y$. Calculate its surface area.



Exercise 11. Solve for x and y from

$$\begin{cases} 2x + y = 7 \\ 5x - 3y = 1 \end{cases}$$

Exercise 12. Solve for x and y:

$$\begin{cases} x = y^2 + 1 \\ y = x - 3 \end{cases}$$

Exercise 13. Solve the following equation for t:

$$\frac{e^t}{1 + e^{2t}} = 0.3$$

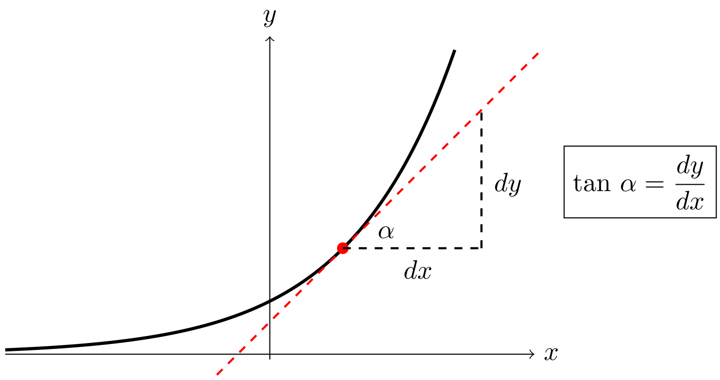
CHAPTER 9

Differentiation

Derivative. The extent to which the graph of $y = f(x)$ increases in a given point is called the 'slope' or 'derivative', for which we use the notation

$$\frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

This is the slope of the tangent line at this point, which is the tangent of the angle α with the 'horizontal'. Calculating the slope is what we call differentiation. In this chapter we learn how to do this.



Standard derivatives. Find below a list of simple functions and their derivatives:

(9.1)

Function	Derivative	Additional information
c	0	The graph of a constant function doesn't tilt.
$ax+b$	a	You should know this from chapter 5.
x^c	$c\,x^{c-1}$	if c is a constant
c^x	$c^x \ln c$	if c is a constant
$\ln x$	$\dfrac{1}{x}$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\dfrac{1}{\cos^2 x}$	

Example 1. The derivative of x^7 is $7x^6$. (Refer to the formula for the derivative of x^c.)

Example 2. The derivative of e^x is e^x. (Refer to the formula for the derivative of c^x or to chapter 6.)

Example 3. The derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$. (Refer to x^c where you substitute the number -1 for c.)

Example 4. The derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$. (Refer to x^c where you substitute the number $\frac{1}{2}$ for c.)

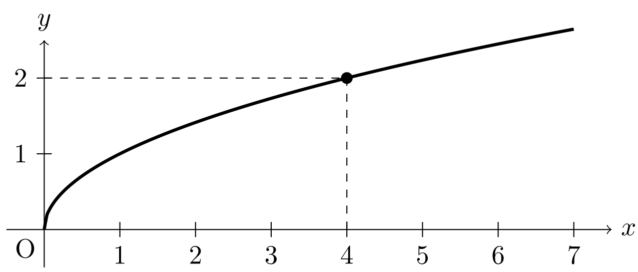
Calculation of the slope. If you want to find the steepness of $y = \sqrt{x}$ in the point $(4, 2)$, your calculation requires two steps:

Step 1. The derivative is $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Step 2. Substitute $x = 4$:

$$\left[\frac{d}{dx}\sqrt{x}\right]_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

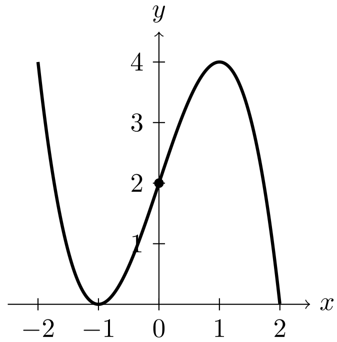
So the slope in the point $(4, 2)$ is $\frac{1}{4}$.



Sum rule. The derivative of the sum is the sum of the derivatives. Finding the derivative of a difference or multiplication by a constant is equally simple:

	Calculation rule	Example
(9.2)	$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$	$\frac{d}{dx}\left(x^3+x^2\right) = 3x^2+2x$
	$\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$	$\frac{d}{dx}\left(x-2^x\right) = 1-2^x\ln 2$
	$\frac{d}{dx}\left(c\cdot f(x)\right) = c\cdot\frac{d}{dx}f(x)$	$\frac{d}{dx}\left(7\sin x\right) = 7\cos x$

Example 5. Determine the equation of the tangent line in $(0,2)$ for the graph of $y = 2+3x-x^3$.



Solution. Let's start by calculating the slope in this point:

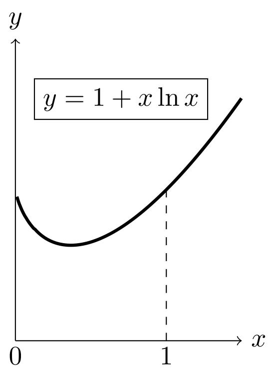
$$\frac{dy}{dx} = 3-3x^2 \quad\Longrightarrow\quad \left[\frac{dy}{dx}\right]_{x=0} = 3$$

So the tangent line is of the type $y = 3x+b$, and by substituting $(0,2)$ you discover that $b = 2$. So the tangent line is $y = 3x+2$.

Product rule. Unfortunately, the derivative of a product is not the product of the derivatives. There is a slightly more complicated calculation rule for this:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x) \tag{9.3}$$

For example: $\frac{d}{dx}\left(x^3\sin x\right) = x^3\cdot\frac{d}{dx}\sin x + (\sin x)\cdot\frac{d}{dx}x^3 = x^3\cos x + 3x^2\sin x$



Example 6. Calculate the minimum of $1+x\ln x$ for $0<x<1$.

Solution. A brilliant idea: where the function is minimal, its slope is zero. Its slope is

$$\frac{d}{dx}\left(1+x\ln x\right) = 0+\frac{d}{dx}x\ln x = x\cdot\frac{d}{dx}\ln x+(\ln x)\cdot\frac{d}{dx}x = 1+\ln x$$

and it is not difficult to determine when this is zero:

$$1+\ln x = 0 \quad\Longrightarrow\quad \ln x = -1 \quad\Longrightarrow\quad x = e^{-1}$$

So the minimum value of the function is $1+e^{-1}\ln e^{-1} = 1-\frac{1}{e}$.

Quotient rule. Taking the derivative of a quotient is not a walk in the park either

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\cdot\frac{d}{dx}f(x) \;-\; f(x)\cdot\frac{d}{dx}g(x)}{\left(g(x)\right)^2} \tag{9.4}$$

For example: $\frac{d}{dx}\frac{2+3x}{x+1} = \frac{(x+1)\cdot 3-(2+3x)\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$

I've always had a hard time memorising this terrible quotient rule, until I found a mnemonic aid in an old Dutch textbook. Unfortunately, I did not manage to find one in English, but I wouldn't dare to keep the Dutch one from you:

$$\frac{d}{dx}\frac{\text{teller}}{\text{noemer}} = \frac{\text{nat}-\text{tan}}{\text{noemer}^2} = \frac{\text{noemer}\cdot\text{afgeleide teller}-\text{teller}\cdot\text{afgeleide noemer}}{\text{noemer}^2}$$

Chain rule. If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{9.5}$$

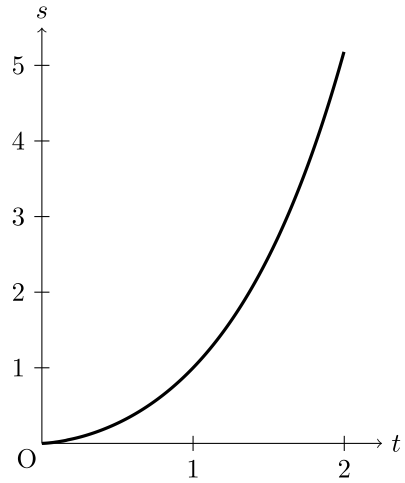
For example: if you want to find the derivative of the function $y = \ln(x^2 + 1)$, you can interpret this as $y = \ln u$ with $u = x^2 + 1$ and apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 2x = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

Example 7. I sprint for two seconds. My displacement $s(t)$ after t seconds for $0 \le t \le 2$ is given by

$$s(t) = \left(2^t - 1\right)^{\frac{3}{2}} \text{ metres}$$

Calculate my velocity at $t = 1$.



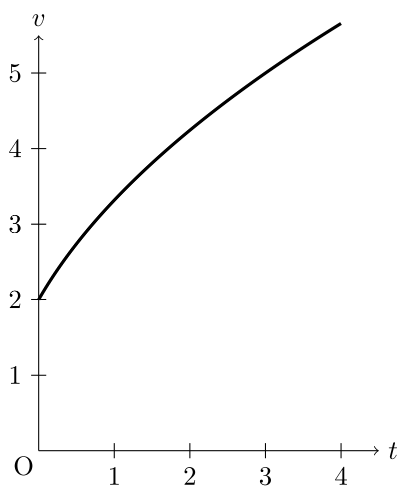
Solution. The velocity $v(t)$ is the increase in displacement per second:

$$v(t) = \frac{d}{dt}\, s(t)$$

I apply the chain rule with $s = u^{\frac{3}{2}}$ and $u = 2^t - 1$:

$$v = \frac{ds}{dt} = \frac{ds}{du} \cdot \frac{du}{dt} = \frac{3}{2} u^{\frac{1}{2}} \cdot 2^t \ln 2 = \frac{3}{2}\sqrt{2^t - 1} \cdot 2^t \ln 2$$

and so $v(1) = 3 \ln 2$ metres per second.



Example 8. (units: kms, hours)
For $0 \le t \le 4$ my walking velocity is given by the formula

$$v(t) = \sqrt{4 + 7t}$$

Calculate my acceleration at $t = 3$.

Solution. The acceleration $a(t)$ is the increase in velocity per hour. According to the chain rule with $v = \sqrt{u}$ and $u = 4 + 7t$ this is

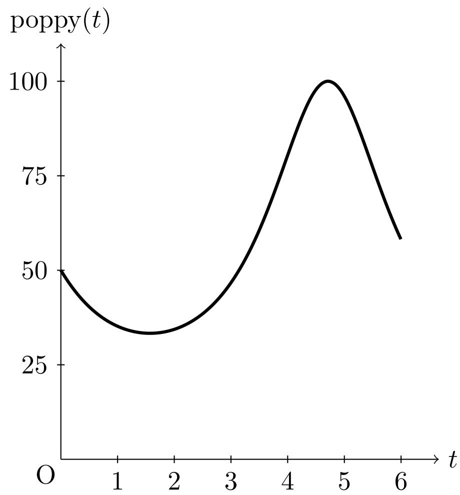
$$a = \frac{dv}{dt} = \frac{dv}{du} \cdot \frac{du}{dt} = \frac{1}{2\sqrt{u}} \cdot 7 = \frac{7}{2\sqrt{4+7t}}$$

At $t = 3$ this is $a(3) = \dfrac{7}{2\sqrt{4+21}} = 0.7 \text{ km/h}^2$.

Example 9. The number of poppies after t years is given by

$$\text{poppy}\,(t) = \frac{100}{2 + \sin t}$$

Calculate the growth of the poppy population at $t = \pi$.



Solution. $\text{poppy} = 100u^{-1}$ where $u = 2 + \sin t$, so

$$\frac{d\,\text{poppy}}{dt} = \frac{d\,\text{poppy}}{du} \cdot \frac{du}{dt} = -100u^{-2} \cdot \cos t = \frac{-100 \cos t}{(2 + \sin t)^2}$$

$$\Longrightarrow \quad \left[\frac{d\,\text{poppy}}{dt}\right]_{t=\pi} = 25 \text{ poppies per year}$$

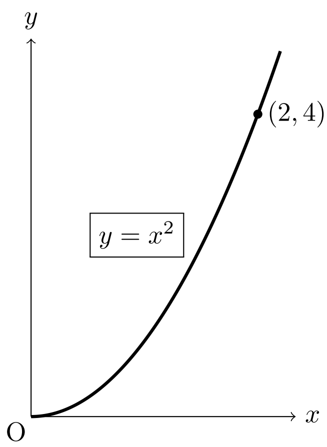
Exercises chapter 9

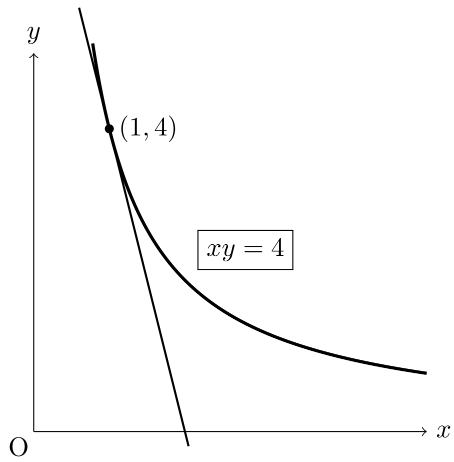
Exercise 1. Calculate the derivative of $\sqrt[3]{x}$.

Exercise 2.

a) Calculate the derivative of $y = x^2$.

b) The graph of $y = x^2$ is pictured in the figure below. Calculate the slope of this graph in the point $(2, 4)$.





Exercise 3. The curve $xy = 4$ is pictured in the figure above. Determine the equation of the tangent line to this curve in the point $(1, 4)$.

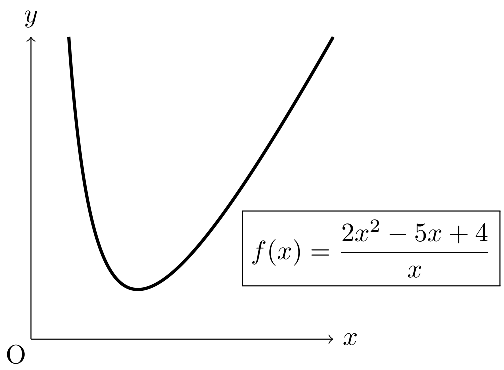
Exercise 4. Find the derivative of

a) $6x\sqrt[3]{x}$ b) $\dfrac{x^2}{7} - \dfrac{7}{x^2}$ c) $\dfrac{x^5 - 3}{x}$

Exercise 5. Consider the function

$$f(x) = \frac{2x^2 - 5x + 4}{x}$$

In my sketch of the graph of f below, you can clearly see that f has a minimum value. Calculate this minimum value.



Exercise 6. Calculate the derivative of

a) $\dfrac{x^2 - 1}{x^2 + 1}$ b) $(1 + 3x)^5$ c) $\ln(1 + \sqrt{x})$

Exercise 7. The distance run by a sprinting athlete after t seconds (in metres) is given by the formula

$$s(t) = \sqrt{t^3 + t^2}$$

Give a function rule for their velocity.

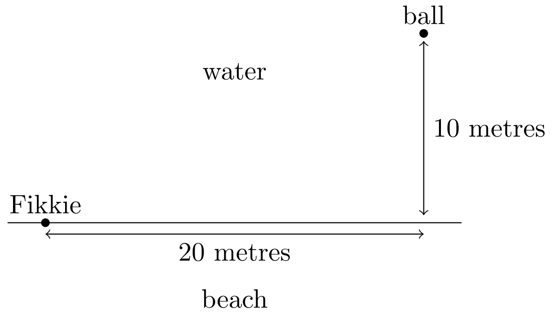
Exercise 8. My cat's weight W (in kg) at time t (in years) is given by

$$W(t) = \ln(1 + t^2)$$

a) Find a formula for the rate at which my cat's weight increases in kg/year.

b) After how many years will my cat be gaining weight fastest?

Exercise 9. I command my dog Fikkie to fetch its ball from the water as soon as possible. Fikkie's running speed is 3 m/sec and its swimming speed is 2 m/sec. How many metres should Fikkie run before jumping into the water?



Exercise 10. Calculate the derivatives of the following functions. Some will be quite challenging!

a) $x \sin x$

b) $\dfrac{1+3x}{1+x}$

c) xe^x

d) $\sin \sqrt{x}$

e) $\dfrac{4x-2}{x^2+1}$

f) $(\sin x + \cos x)^3$

g) $\sin(x + e^x)$

h) $\ln(x + \sin x)$

i) $\ln(1 + \cos^2 x)$

j) $e^{(x^3)}$

CHAPTER 10

Antidifferentiation

Antidifferentiation. In the previous chapter we have seen that a function $f(x)$ can be differentiated to its so-called derivative function $f'(x)$, which describes the slope of $f(x)$ at any point x. Sometimes, however, we are interested in the exact opposite action: given the derivative function, we would like to know what the original function is. Finding the original function from its derivative is what we call antidifferentiation.

Notation. Antidifferentiation is reverse differentation: you are looking for the original function that, when differentiated, yields your function of interest $f(x)$. The original function is called the antiderivative (or the indefinite integral or the primitive function) of $f(x)$ and is usually denoted with the a capital letter: $F(x)$. Thus, $F(x)$ is an antiderivative of $f(x)$ if the derivative of $F(x)$ equals $f(x)$:

$$\boxed{F'(x) = f(x)} \quad \text{or, put differently,} \quad \boxed{\int f(x)\,dx = F(x)}$$

The integration sign $\int$ (without values attached to it) just means 'the antiderivative of', and dx means that you are antidifferentiating with respect to the variable x. Don't forget to include this 'd-something', especially if you are antidifferentiating a function containing multiple variables.

Example 1. Let's look at the function $f(x) = 2x$. Find an antiderivative of $f(x)$.

Solution. We are looking for a function $F(x)$ whose slope is the linear function $f(x) = 2x$. You might remember from high school that linear slopes are indicative of second degree polynomials, and you should remember from the previous chapter that $\frac{d}{dx}\,x^2 = 2x$. Therefore, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$:

$$\int 2x\,dx = x^2$$

Standard antiderivatives. Since antidifferentation is the reverse of differentiation, we can construct a list of standard antiderivatives by reversing the columns from our list of standard derivatives and making some cosmetic adjustments:

(10.1)

Function	Antiderivative	Additional information
0	C	with C an arbitrary constant
a	ax	if a is a constant
x^a	$\frac{1}{a+1}\,x^{a+1}$	if a is a constant $\neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\lvert x\rvert$ because ln(negative) does not exist
a^x	$\frac{1}{\ln a}\,a^x$	if a is a positive constant
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	

The antiderivative of zero. The first rule of our list of standard antiderivatives might surprise you: how can the antiderivative of zero be *any* constant number? Well, as you remember, constants vanish upon differentiation, so you can always add any constant number to an antiderivative. This is why we often put '$+\,C$' behind a calculated antiderivative, with the constant C representing an arbitrary real number. This is usually a formality rather than a rule, and many people just forget about these constants. Most often you will be excused if you do too, until you start working with initial value problems or differential equations, when adding a constant becomes a very essential step. This is why it is probably smart to teach yourself to always add a constant to an antiderivative.

Example 2. Calculate the antiderivative of $f(x) = 8x^3$.

Solution. We could use a standard antiderivative to solve this problem:

$$\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}$$

However, let's try to see if we can argue what the antiderivative of $f(x) = 8x^3$ should be, which will hopefully help us understand where this standard antiderivative comes from. Let's look back at what we learned about differentiating powers of x: $\frac{d}{dx}x^c = cx^{c-1}$. If we put this rule into words, we have: in order to differentiate a power of x you should first <u>multiply the function by the exponent</u> and then <u>decrease the exponent by 1</u>.

Now, let's reverse this process: in order to antidifferentiate a power of x you should <u>increase the exponent by 1</u> and <u>divide the function by this new exponent</u>. So we have:

$$8x^3 \xRightarrow{\text{increase exponent by 1}} 8x^4 \xRightarrow{\text{divide by new exponent}} 2x^4$$

Conclusion: if $f(x) = 8x^3$, then $F(x) = 2x^4 + C$. You can check that this antiderivative is correct by differentiating it.

If we generalise the procedure that we just followed, we obtain the standard antiderivative of x^a:

$$x^a \xRightarrow{\text{increase exponent by 1}} x^{a+1} \xRightarrow{\text{divide by new exponent}} \frac{1}{a+1}\,x^{a+1}$$

The other standard antiderivatives can be obtained in similar ways, or you can just prove that they are correct by differentiating them.

Example 3. Calculate:

a) $\displaystyle\int 9x^6\,dx$ b) $\displaystyle\int \frac{18}{x^4}\,dx$ c) $\displaystyle\int \sqrt{x}\,dx$ d) $\displaystyle\int \frac{36x^4}{\sqrt{x}}\,dx$ e) $\displaystyle\int \frac{1}{x}\,dx$

Solution. The first four of these antiderivatives can be tackled using our standard antiderivative for powers of x by rewriting the function to be antidifferentiated as x^a:

a) $\displaystyle\int 9x^6\,dx = \frac{9}{7}x^7 + C$

b) $\displaystyle\int \frac{18}{x^4}\,dx = \int 18x^{-4}\,dx = \frac{18}{-3}\,x^{-3} + C = -\frac{6}{x^3} + C$

c) $\displaystyle\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{1}{3/2}\,x^{\frac{3}{2}} + C = \frac{2}{3}\,x\sqrt{x} + C$

d) $\displaystyle\int \frac{36x^4}{\sqrt{x}}\,dx = \int \frac{36x^4}{x^{\frac{1}{2}}}\,dx = \int 36x^{\frac{7}{2}}\,dx = \frac{36}{9/2}\,x^{\frac{9}{2}} + C = 8x^4\sqrt{x} + C$

However, if you try to apply this rule to find the antiderivative of $\frac{1}{x}$, you will run into bad trouble, because

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \frac{1}{0}\,x^0$$

obviously doesn't make any sense. Instead, you should just take the standard antiderivative dedicated to $\frac{1}{x}$:

e) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

Example 4. Calculate $\displaystyle\int e^x\,dx$.

Solution. You can use the standard antiderivative of a^x with $a = e$:

$$\int e^x\,dx = \frac{1}{\ln e}\,e^x + C = \frac{1}{1}\,e^x + C = e^x + C$$

Conclusion: the antiderivative of e^x is just e^x (plus an arbitrary constant), which shouldn't surprise you since you have met the special property of e^x in chapter 6.

Calculation rules for antiderivatives. The antiderivative of the sum is the sum of the antiderivatives, which follows immediately from the sum rule for differentiation. The difference of two antiderivatives or an antiderivative multiplied by a constant can also be evaluated easily using the differentiation rules from the previous chapter:

	Calculation rule	Example
(10.2)	$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$	$\int (x^3+x^2)\,dx = \frac{x^4}{4}+\frac{x^3}{3}+C$
	$\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$	$\int (x-2^x)\,dx = \frac{x^2}{2}-\frac{2^x}{\ln 2}+C$
	$\int (c\cdot f(x))\,dx = c\cdot\int f(x)\,dx$	$\int (7\sin x)\,dx = -7\cos x + C$

However, there exists no such thing as the product rule or the quotient rule for antiderivatives! Antidifferentiation of product functions or quotient functions can be a complicated task, and there exist many functions that just cannot be antidifferentiated at all.

Example 5. Calculate:

a) $\int (\sin x + \cos x)\,dx$ b) $\int \left(\frac{3}{x^2}-\frac{x^2}{3}\right)dx$ c) $\int \left(\frac{3}{x}-\frac{x}{3}\right)dx$ d) $\int (2^x+4^x)\,dx$

Solution.

a) $\int (\sin x+\cos x)\,dx = \int \sin x\,dx + \int \cos x\,dx = -\cos x + \sin x + C$
Note that there's no need to add *two* arbitrary constants C_1 and C_2, because we can just combine them into one constant $C = C_1 + C_2$.

b) $\int \left(\frac{3}{x^2}-\frac{x^2}{3}\right)dx = \int 3x^{-2}\,dx - \int \frac{x^2}{3}\,dx = 3\int x^{-2}\,dx - \frac{1}{3}\int x^2\,dx = 3\cdot -x^{-1} - \frac{1}{3}\cdot\frac{x^3}{3} + C = -\frac{3}{x} - \frac{x^3}{9} + C$

c) $\int \left(\frac{3}{x}-\frac{x}{3}\right)dx = \int \frac{3}{x}\,dx - \int \frac{x}{3}\,dx = 3\ln x - \frac{x^2}{6} + C$

d) $\int (2^x+4^x)\,dx = \int 2^x\,dx + \int 4^x\,dx = \frac{2^x}{\ln 2} + \frac{4^x}{\ln 4} + C = \frac{2^x}{\ln 2} + \frac{4^x}{2\ln 2} + C = \frac{2^{x+1}+4^x}{2\ln 2} + C$

Example 6. What is $\int x\sin x\,dx$?

Solution. $f(x) = x\sin x$ is a perfect example of a product function which you can differentiate very easily using the product rule, but which is quite hard to be antidifferentiated. There exists no rule to antidifferentiate product functions, but what we can do is just give it a shot and see what happens if we take $F(x) = x\cos x$ as a trial antiderivative. Let's differentiate this $F(x)$:

$$\frac{d}{dx}F(x) = \frac{d}{dx}x\cos x = \cos x - x\sin x$$

This is almost equal to our original function $f(x)$, but we have two little problems to solve:

- There is an extra term $\cos x$ that we should get rid of, but this should not be too difficult: we can just add a term $-\sin x$ to our antiderivative, because this will yield $-\cos x$ upon differentiation.
- There is a minus sign in front of $x\sin x$, but we can just multiply our antiderivative by -1 to get rid of this minus sign.

Conclusion: $\int x\sin x\,dx = \sin x - x\cos x + C$.

You might be a bit disappointed by the approach we used here: 'giving it a shot' and 'seeing what happens' are not very rigorous. There exist techniques to tackle these kind of more complicated antiderivatives and you will learn some of them in your mathematics courses, but you will hopefully also get used to the fact that antidifferentiation remains a bit of a trial-and-error process.

Reverse chain rule. We've still got one important differentation rule left to discuss in the context of antidifferentiation and that is the chain rule. The chain rule for differentiation says:

> If y is a function of u, and u is a function of x, then the derivative of y with respect to x is equal to:
> - the derivative of y with respect to u
> - multiplied by the derivative of u with respect to x

The essential step here is the second, in which you multiply by the derivative of u with respect to x: this step is absent when differentiating 'normal' functions. Now, if we reverse the chain rule for differentiation to make it applicable to antidifferentiation, we will have to compensate for this extra multiplication step. Our first guess could be the following:

If y is a function of u, and u is a function of x, then the *antiderivative* of y with respect to x is equal to:
- the *antiderivative* of y with respect to u
- *divided* by the derivative of u with respect to x

Let's check if this reverse chain rule works:

Example 7. Calculate $\int (1-3x)^5\,dx$.

Solution. We have $y = u^5$ and $u = 1-3x$, so the antiderivative of y with respect to u is

$$\int u^5\,du = \frac{u^6}{6} = \frac{1}{6}(1-3x)^6$$

and the derivative of u with respect to x is -3. Now, according to our guessed chain rule for antidifferentiation, the resulting antiderivative is

$$\int (1-3x)^5\,dx = \frac{\frac{1}{6}(1-3x)^6}{-3} + C = -\frac{1}{18}(1-3x)^6 + C$$

Let's check what happens if we differentiate this antiderivative:

$$\frac{d}{dx}\left(-\frac{1}{18}(1-3x)^6 + C\right) = -\frac{6}{18}(1-3x)^5 \cdot (-3) = (1-3x)^5$$

Conclusion: our guessed reverse chain rule seems to work. However, you will see that we run into problems if we take another example:

Example 8. What is $\int \exp(x^2)\,dx$?

Solution. In this case we have $y = e^u$, whose antiderivative is $\int e^u\,du = e^u$, and we have $u = x^2$, whose derivative is $\frac{du}{dx} = 2x$. According to our tentative chain rule, the total antiderivative is

$$\frac{e^u}{2x} + C = \frac{\exp(x^2)}{2x} + C$$

but let's check what happens if we differentiate this function: we will have to use the quotient rule for this, so

$$\frac{d}{dx}\left(\frac{\exp(x^2)}{2x} + C\right) = \frac{2x \cdot 2x\exp(x^2) - 2\exp(x^2)}{4x^2} = \exp(x^2) - \frac{\exp(x^2)}{2x^2}$$

The first term looks promising, but it is impossible to get rid of the second term. In fact, it is just not possible to calculate the antiderivative of $\exp(x^2)$ at all. Clearly, our tentative reverse chain rule doesn't work in all cases. The crux is that our rule only works if the derivative of u with respect to x is a constant, so that you should divide the antiderivative of y with respect to u by a constant. $\frac{du}{dx}$ is a constant if u is a linear function of x. Therefore:

(10.3)

> <u>Reverse chain rule for antidifferentation</u>
>
> If y is a function of u, and u is a **linear** function of x, then the *antiderivative* of y with respect to x is equal to:
> - the *antiderivative* of y with respect to u
> - *divided* by the derivative of u with respect to x, which should be a constant

If u is not a linear function of x, you can sometimes calculate the antiderivative of y using other techniques, which you will learn in your mathematics courses.

Exercises chapter 10

Exercise 1. Calculate:

a) $\displaystyle\int 4x^2\,dx$ b) $\displaystyle\int \frac{5}{8}x^{10}\,dx$ c) $\displaystyle\int \frac{7x^7}{3x^3}\,dx$ d) $\displaystyle\int \sqrt{2x}\,dx$ e) $\displaystyle\int \frac{5}{4x^2}\,dx$

Exercise 2. Calculate:

a) $\displaystyle\int 3\sqrt{x+5}\,dx$ b) $\displaystyle\int \frac{1}{\sqrt{3t+2}}\,dt$ c) $\displaystyle\int \sqrt{13-\alpha}\,d\alpha$

Exercise 3. Calculate:

a) $\displaystyle\int e^{3x}\,dx$ b) $\displaystyle\int \frac{2}{e^{2t}}\,dt$ c) $\displaystyle\int \frac{e^x-e^{-x}}{2}\,dx$ d) $\displaystyle\int \frac{5}{6y}\,dy$

Exercise 4. Calculate:

a) $\displaystyle\int \cos(\pi x)\,dx$ b) $\displaystyle\int \sin(ax+b)\,dx$ with a and b constants c) $\displaystyle\int \sqrt{2}\cos(1-x)\,dx$

Exercise 5. Calculate:

a) $\displaystyle\int 2\sin x\cos x\,dx$ b) $\displaystyle\int \left(2x\sin^2 x+2x\cos^2 x\right)dx$ c) $\displaystyle\int 3\cos x\tan x\,dx$

Exercise 6. Find the antiderivative of $f(x)=\ln x$. *Hint*: use the 'give it a shot and see what happens' approach by starting with $F(x)=x\ln x$ and adjusting it until it is indeed an antiderivative of $f(x)=\ln x$.

Exercise 7. Given is the family of functions

$$f_p(x)=\frac{4(x+9)(x+p)}{x^2+12x+27}$$

Calculate $\displaystyle\int f_3(x)\,dx$.

Exercise 8. In Example 8 from the previous chapter I went for a walk with my walking velocity v as a function of the time t being given by

$$v(t)=\sqrt{4+7t}$$

What was the distance I covered in 3 hours?

Exercise 9.

a) Simplify $f(x)=\dfrac{3}{x^2-1}$ using partial fraction decomposition and calculate its antiderivative.

b) Simplify $f(x)=\dfrac{2x^3-x+1}{x-3}$ using long division and calculate its antiderivative.

CHAPTER 11

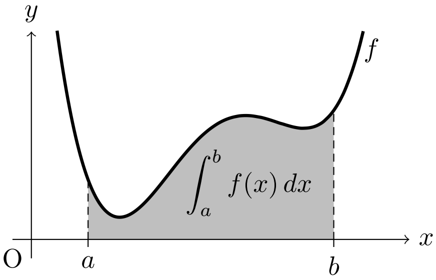
Integration

Fundamental theorem of calculus. The fundamental theorem of calculus (FTOC), which is even more important than its name already suggests (in Dutch: "Hoofdstelling der integraalrekening"), says that the area enclosed by the graph of a function $f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\boxed{\begin{array}{c}\text{area enclosed by the graph of } f\text{, the}\\ x\text{-axis and the lines } x=a \text{ and } x=b\end{array}} = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \tag{11.1}$$

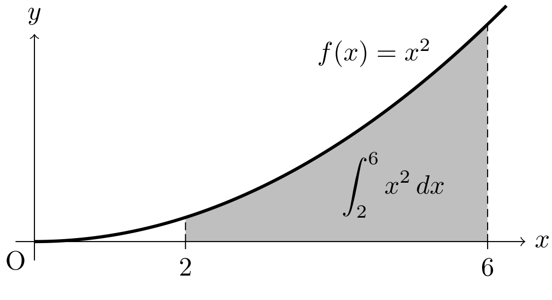
where $F(x)$ is (as you should know after the previous chapter) an antiderivative of $f(x)$. Let's discuss the different elements of the FTOC step by step:

- $\boxed{\begin{array}{c}\text{area enclosed by the graph of } f\text{, the}\\ x\text{-axis and the lines } x=a \text{ and } x=b\end{array}}$: let's draw a picture to illustrate what we mean by this area:



- $\displaystyle\int_a^b f(x)\,dx$ is pronounced as 'the (definite) integral of f from $x = a$ to $x = b$'.
  This thing obviously looks quite similar to the antiderivative of f, which is $\int f(x)\,dx$: the only difference is that we have attached the two integration limits a and b to the integration sign $\int$. These limits tell us that we not only have to *calculate* the antiderivative, but also *evaluate* it at $x = a$ and $x = b$.
- $F(b) - F(a)$: this is exactly where we evaluate the antiderivative at $x = a$ and $x = b$. The FTOC says that we first have to evaluate $F(x)$ at $x = b$ and then subtract the value of $F(x)$ at $x = a$ to yield the desired area.
- $\Big[F(x)\Big]_a^b$ is just a convenient shorthand notation for $F(b) - F(a)$.

Example 1. Calculate the area between the graph of $f(x) = x^2$, the x-axis and the lines $x = 2$ and $x = 6$:



Solution. This is exactly the kind of problem that we can solve with the FTOC:

$$\text{area} = \int_2^6 f(x)\,dx = \Big[F(x)\Big]_2^6 = F(6) - F(2)$$

An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, so

$$\text{area} = \int_2^6 x^2\,dx = \left[\frac{1}{3}x^3\right]_2^6 = \frac{1}{3}\left(6^3 - 2^3\right) = \frac{1}{3}\left(216 - 8\right) = \frac{208}{3}$$

Note that we didn't add an arbitrary constant C to the antiderivative, because this constant would have disappeared anyway when subtracting $F(2)$ from $F(6)$.

Let's summarise the procedure that we followed:

(11.2)

<u>How to calculate the integral of f from a to b</u>

1. Calculate an antiderivative F of f.
2. Substitute $x = b$ into $F(x)$.
3. Substitute $x = a$ into $F(x)$ and subtract this from $F(b)$.

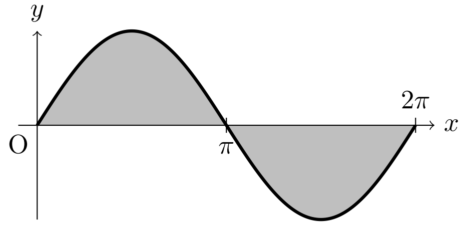
$$\int_a^b f(x)\,dx$$

Example 2. Calculate the area A between the graph of $f(x) = e^x$ and the x-axis from $x = 2$ to $x = 4$.

Solution. With the FTOC under your belt nothing can stop you from tackling this problem:

$$A = \int_2^4 f(x)\,dx = \int_2^4 e^x\,dx = \Big[e^x\Big]_2^4 = e^4 - e^2 = \left(e^2\right)^2 - e^2 = e^2\left(e^2 - 1\right)$$

Example 3. Calculate the area A between the graph of $f(x) = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$.



Solution.

$$\int_0^{2\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{2\pi} = -\cos 2\pi - (-\cos 0) = -1 - (-1) = 0$$

Our integral returns zero, but the desired area is clearly nonzero: what just happened? Well, the graph of $f(x) = \sin x$ intersects the x-axis on the domain $0 \leq x \leq 2\pi$, and we haven't discussed yet how you should handle areas located under the x-axis. We will have to make use of the following fact:

(11.3)

If $f(x) \leq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx = -\int_a^b |f(x)|\,dx$.

In words: areas located under the x-axis count as negative in integrals.

This fact explains why our integral returned zero: the part of the area above the x-axis is exactly equal to the part of the area below the x-axis, so when calculating the integral, both parts cancel and we obtain zero.

Now, there are two ways to calculate the actual value of the total area A:

- The general approach is to split up the integral into different parts, then calculate the area of each part separately and finally add up all parts to obtain the total area *taking into account the signs of the area parts.* In this case, we have:

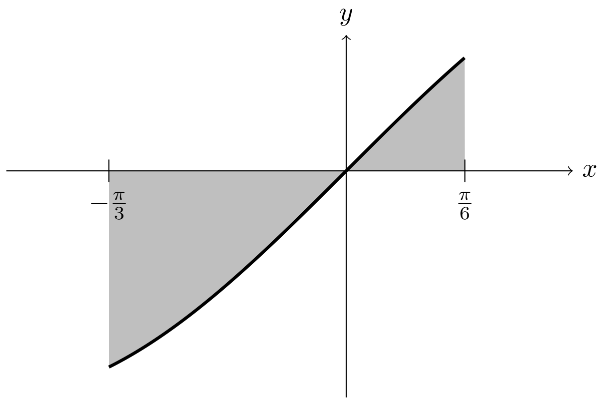
$$\int_0^{\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$$
$$\int_{\pi}^{2\pi} \sin x\,dx = \Big[-\cos x\Big]_{\pi}^{2\pi} = -\cos 2\pi - (-\cos\pi) = -1 - 1 = -2$$

$$A = 2 - (-2) = 4$$

- Sometimes you can conveniently use the symmetry of the problem: in this case, both parts of the total area are clearly equally big, so we could also calculate only the left part and then multiply the result by 2 to obtain the total area:

$$A = 2\int_0^{\pi} \sin x\,dx = 2\Big[-\cos x\Big]_0^{\pi} = 2\cdot(-\cos\pi - (-\cos 0)) = 2\cdot(1 - (-1)) = 4$$

Example 4. In the USA, there are four states that meet in a perfectly square quadripoint $(0, 0)$; Arizona $(x < 0, y < 0)$, Utah $(x < 0, y > 0)$, Colorado $(x > 0, y > 0)$ and New Mexico $(x > 0, y < 0)$ (don't hesitate to Google it). Arizona and Colorado are fighting over area of a river belonging to their territory. The river is called Sine River, and its name originates from its distinctive shape: it flows according to $f(x) = \sin(x)$. Arizona's government has dictated that people can only live north of the river (see the shaded area in the lower left corner of the figure below), whereas in Colorado people are only allowed to live south of the river (shaded area in the upper right corner). What is the total liveable area near the river, and how much more area does Arizona get?



Solution. We are interested in the total liveable area, so we should split the integral into two parts:

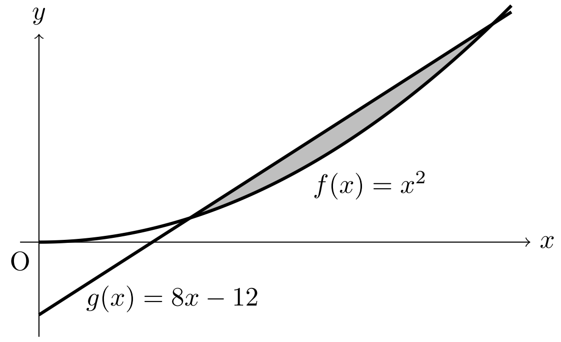
$$A_{\text{Arizona}} = \left| \int_{-\frac{\pi}{3}}^{0} \sin(x)\, dx \right| = \left| [-\cos(x)]_{-\frac{\pi}{3}}^{0} \right| = \left| -\cos(0) + \cos\left(-\frac{\pi}{3}\right) \right| = \left| -1 + \frac{1}{2} \right| = \frac{1}{2}$$

$$A_{\text{Colorado}} = \int_{0}^{\frac{\pi}{6}} \sin(x)\, dx = [-\cos(x)]_{0}^{\frac{\pi}{6}} = -\cos\left(\frac{\pi}{6}\right) + \cos(0) = 1 - \frac{\sqrt{3}}{2}$$

Conclusion: the total area equals $\frac{1}{2} + 1 - \frac{\sqrt{3}}{2} = \frac{3 - \sqrt{3}}{2}$

To find out how much more area was assigned to Arizona, you can just take the entire integral from $-\frac{1}{3}\pi$ to $\frac{1}{6}\pi$ (without splitting and taking absolute values): this turns out to be $-\frac{1}{2}(\sqrt{3} - 1)$ (which you should definitely check for yourself), so Arizona has $\frac{1}{2}(\sqrt{3} - 1)$ more square miles of land.

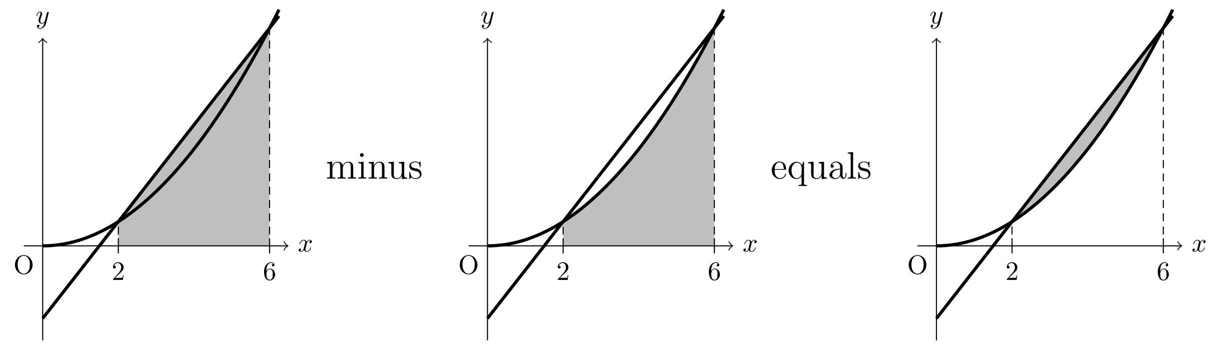
Example 5. Calculate the area A enclosed by the graphs of $f(x) = x^2$ and $g(x) = 8x - 12$.



Solution. Let's first find out where the two graphs intersect:

$$x^2 = 8x - 12 \iff x^2 - 8x + 12 = (x - 4)^2 - 4 = 0 \iff x - 4 = \pm 2 \iff x = 2 \text{ and } x = 6$$

From Example 1 we know that the area between the graph of $f(x) = x^2$ *and the x-axis* from $x = 2$ to $x = 6$ is equal to $\frac{208}{3}$, but here we are interested in the small part of area *above* the graph of $f(x) = x^2$ and *below* the graph of $g(x) = 8x - 12$. This area is completely different from what we calculated in Example 1, but we can make clever use of the result of Example 1. If you look at the figures on the next page, you can see how: if we first calculate the area between the graph of g and the x-axis and then subtract the area between f and the x-axis, which is the result from Example 1, we obtain the desired area A:



The actual calculations goes as follows:

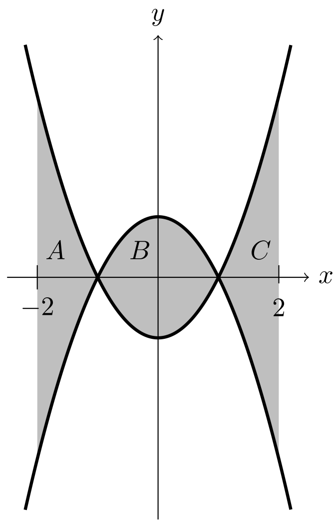
$$A = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx = \int_2^6 (8x-12)\,dx - \frac{208}{3} = \frac{1}{16}\Big[(8x-12)^2\Big]_2^6 - \frac{208}{3}$$
$$= \frac{1}{16}(36^2 - 4^2) - \frac{208}{3} = \frac{1}{16}(9^2-1)\cdot 4^2 - \frac{208}{3} = 80 - \frac{208}{3} = \frac{32}{3}$$

We calculated the integrals of f and g separately in this example because we could use Example 1, were we had already calculated the integral of f. However, it is often more convenient to combine f and g into a single integral as follows:

(11.4) $$\boxed{\begin{array}{c}\text{area enclosed by the graphs of } f \text{ and } g\\ \text{and the lines } x=a \text{ and } x=b\\ \text{if } g(x)\geq f(x) \text{ on } a\leq x\leq b\end{array}} = \int_a^b (g(x)-f(x))\,dx$$

Conclusion: if you want to calculate the area between the graphs of two functions, you should calculate the integral of the upper function minus the lower function.

Example 6. Calculate the total enclosed area between $f(x) = 1 - x^2$ and $g(x) = x^2 - 1$ in the region from $x = -2$ to $x = 2$. The graphs intersect at $x = -1$ and $x = 1$.



Solution. You can see that we will have to split up our integral into three parts: in region A we have $g(x) > f(x)$, in region B we have $f(x) > g(x)$, and in region C we have $g(x) > f(x)$ again:

$$A = \int_{-2}^{-1} (g(x)-f(x))\,dx = \int_{-2}^{-1} \big(x^2 - 1 - (1-x^2)\big)\,dx = \int_{-2}^{-1} \big(2x^2 - 2\big)\,dx = \Big[\frac{2}{3}x^3 - 2x\Big]_{-2}^{-1} = \cdots = \frac{8}{3}$$
$$B = \int_{-1}^{1} (f(x)-g(x))\,dx = \int_{-1}^{1} \big(1 - x^2 - (x^2-1)\big)\,dx = \int_{-1}^{1} \big(2 - 2x^2\big)\,dx = \Big[2x - \frac{2}{3}x^3\Big]_{-1}^{1} = \cdots = \frac{8}{3}$$
$$C = A = \frac{8}{3} \quad \text{because of the symmetry of the graphs of } f \text{ and } g$$

Conclusion: the total area is 8.

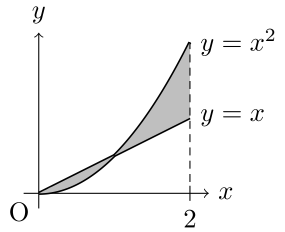
Exercises chapter 11

Exercise 1. Sketch and calculate the integral of $f(x) = \frac{5}{2x}$ from $x = e^2$ to $x = e^3$.

Exercise 2. Calculate the total (absolute) area that is enclosed between the x-axis and the graph of $f(x) = \sin(\frac{1}{2}x)$ for three total periods.

Exercise 3. Calculate the area enclosed by the x-axis and the graph of $f(x) = e^{4x} - 2\sqrt{x}$ from $x = 0$ to $x = 1$.

Exercise 4. Calculate the area of the shaded region in the figure below.



Exercise 5. David, Frank and Sammy are asked to calculate $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

David comes up with the following solution:

$$\int_0^1 \sqrt{1-x^2}\,dx = \left[\frac{2}{3}\cdot\frac{1}{2}\cdot\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}\left(\left(1-1^2\right)^{\frac{3}{2}} - \left(1-0^2\right)^{\frac{3}{2}}\right) = \frac{1}{3}$$

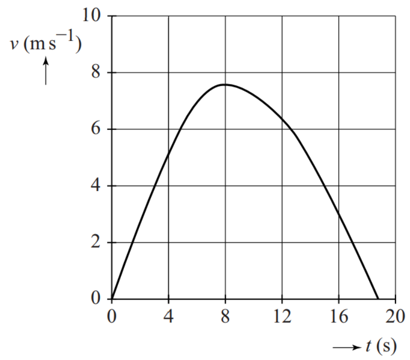
Frank comes up with the following solution:

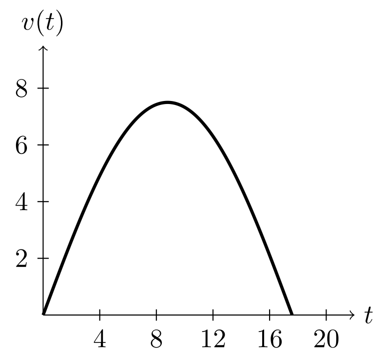
$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^1 \sqrt{(1-x)\,(1+x)}\,dx = \left[-\,(1-x)^{\frac{3}{2}}\,(1+x)^{\frac{3}{2}}\right]_0^1 = 2-2\sqrt{2}$$

Sammy looks at these attempts and says: "You guys are making a lot of errors. I only needed a single second to know the answer without actually trying to calculate the integral: the answer is $\frac{\pi}{4}$."

Explain what mistakes David and Frank made, and what reasoning Sammy used to arrive at his answer.

Exercise 6. One of the exercises of the Dutch 2013 VWO Physics exam was to find the distance covered by a tractor in a tractor pulling event from the (v, t)-diagram (velocity as a function of time) in the figure below. Although the exam candidates were allowed to estimate this distance from the graph, we can actually calculate the distance if we simulate the graph by $v(t) = \frac{15}{2}\sin\left(\frac{5}{28}t\right)$ from $t = 0$ to $t = 18$. Use integration to find the distance that the tractor covered.





Exercise 7. If you take the area under the graph of a function f from x_{left} to x_{right}, and rotate this area over 360° around the x-axis, you obtain a three-dimensional object called the *solid of revolution.* The volume V of this solid of revolution is given by the following integral:

$$V = \int_{x_{\text{left}}}^{x_{\text{right}}} \pi\cdot(f(x))^2\,dx$$

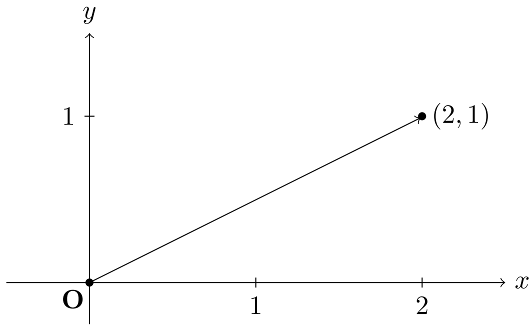
Calculate the volume of the solid of revolution of $f(x) = \dfrac{6}{\sqrt{x}}$ from $x = 1$ to $x = 17$.

CHAPTER 12

Vectors

Vectors and scalars. A vector $\vec{\mathbf{a}}$ is a quantity with a magnitude and a direction. Vectors can be two-dimensional (in $\mathbb{R}^2$), three-dimensional (in $\mathbb{R}^3$) or even n-dimensional (in $\mathbb{R}^n$). Vector quantities differ from scalar quantities, because scalars only have a magnitude without a direction. Examples of vector quantities: force, velocity, magnetic and electric fields. Examples of scalar quantities: mass, energy and temperature. We use boldface letters with an arrow on top to label vectors, so that you never confuse vectors with numbers.

Vectors in $\mathbb{R}^2$. A vector $\vec{\mathbf{a}}$ in $\mathbb{R}^2$ can be represented as a row with two components: (a_x, a_y). The numbers a_x and a_y are called the coordinates or the components of the vector. If you draw a system of axes and choose a unit length, you can represent a vector as an arrow in the plane. For example the vector $(2, 1)$:



Instead of the arrow from $(0, 0)$ to $(2, 1)$ you can also say that the point $(2, 1)$ itself represents the vector. However, this point is still understood to be relative to $(0, 0)$, so the directional aspect of the vector is still contained in the point $(2, 1)$. The origin $(0, 0)$ can also be written as $\mathbf{O}$. In this figure above I've drawn the first coordinate axis horizontally and the second axis vertically. The first coordinate axis is also called the x-axis (or sometimes the x_1-axis), and the second axis the y-axis (or the x_2-axis).

Column vector notation. In some contexts the following notation for a vector is used instead of (a_x, a_y):

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Vectors manipulations. In chapter 1 we have seen that we can do addition, subtraction and multiplication with scalar numbers. These three basic operations are also defined for vectors and they go *component-wise*:

- Adding two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ amounts to adding the x-components of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to obtain the x-component of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, and adding the y-components of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to obtain the y-component of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$.
- Subtracting a vector $\vec{\mathbf{b}}$ from another vector $\vec{\mathbf{a}}$ amounts to subtracting the x-component of $\vec{\mathbf{b}}$ from the x-component of $\vec{\mathbf{a}}$ to obtain the x-component of $\vec{\mathbf{a}} - \vec{\mathbf{b}}$, and subtracting the y-component of $\vec{\mathbf{b}}$ from the y-component of $\vec{\mathbf{a}}$ to obtain the y-component of $\vec{\mathbf{a}} - \vec{\mathbf{b}}$.
- Multiplication of a vector $\vec{\mathbf{a}}$ by a scalar number λ amounts to multiplying the x-component of $\vec{\mathbf{a}}$ by λ to obtain the x-component of $\lambda \cdot \vec{\mathbf{a}}$, and multiplying the y-component of $\vec{\mathbf{a}}$ by λ to obtain the y-component of $\lambda \cdot \vec{\mathbf{a}}$.

Note that these three basic vector operations all yield a new vector: the sum of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is another vector $\vec{\mathbf{a}}+\vec{\mathbf{b}}$, the difference of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is another vector $\vec{\mathbf{a}}-\vec{\mathbf{b}}$, and the scalar product of $\vec{\mathbf{a}}$ by λ is another vector $\lambda \cdot \vec{\mathbf{a}}$.

There also exist vector manipulations that yield a scalar number instead of another vector:

- The dot product $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}$ (also called scalar product or inner product) of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ yields a scalar number.
- The magnitude $|\vec{\mathbf{a}}|$ (also called length or norm) of a vector $\vec{\mathbf{a}}$ is a scalar number.
- The angle φ between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is a scalar number.

These three manipulations are defined in the box with calculations rules on the next page.

Calculation rules for vectors.

(12.1)

Vector addition (componentwise):	$(a_x, a_y) + (b_x, b_y) = (a_x + b_x, a_y + b_y)$
Vector subtraction (componentwise):	$(a_x, a_y) - (b_x, b_y) = (a_x - b_x, a_y - b_y)$
Scalar multiplication (componentwise):	$\lambda \cdot (a_x, a_y) = (\lambda \cdot a_x, \lambda \cdot a_y)$
Dot product:	$(a_x, a_y) \bullet (b_x, b_y) = a_x b_x + a_y b_y$
Magnitude:	$\lvert(a_x, a_y)\rvert = \sqrt{a_x^2 + a_y^2}$
Angle:	$\cos\varphi = \dfrac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{\lvert\vec{\mathbf{a}}\rvert \cdot \lvert\vec{\mathbf{b}}\rvert}$

Example 1. Addition and subtraction go as follows:

$$(2,1) + (-1,1) = (1,2) \qquad (-\pi, \sqrt{7}) + (3,0) = (3-\pi, \sqrt{7}) \qquad \left(\frac{5}{2}, \frac{6}{7}\right) - \left(\frac{7}{3}, \frac{1}{7}\right) = \left(\frac{1}{6}, \frac{5}{7}\right)$$

Example 2. This is how scalar multiplication works:

$$3 \cdot (2\,, -1) = (6\,, -3) \qquad \sqrt{2}\,(7\,, \sqrt{3}) = (7\sqrt{2}\ ,\ \sqrt{6}) \qquad \frac{2}{3}(-12, 5) = \left(-8\,, \frac{10}{3}\right)$$

Example 3. The dot product is the sum of the componentwise multiplication of vectors:

$$(5,-1) \bullet (2,7) = 5 \cdot 2 + (-1) \cdot 7 = 3 \qquad (3, 8\pi) \bullet (-4, 0) = 3 \cdot (-4) + 8\pi \cdot 0 = -12$$

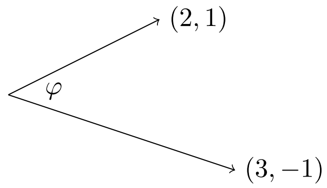
Example 4. The magnitude is the square root of the sum of the squares of the components:

$$\lvert(5,12)\rvert = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \qquad \lvert(-4,11)\rvert = \sqrt{(-4)^2 + 11^2} = \sqrt{137}$$

Example 5. The angle φ between the vectors $(2,1)$ and $(3,-1)$ is given by

$$\cos\varphi = \frac{(2,1) \bullet (3,-1)}{\lvert(2,1)\rvert \cdot \lvert(3,-1)\rvert} = \frac{5}{\sqrt{5} \cdot \sqrt{10}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

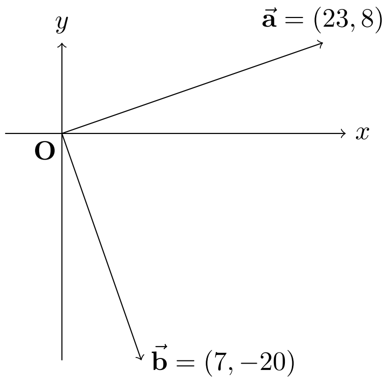
so $\varphi = \dfrac{\pi}{4}$ radians, which is 45 degrees.



Perpendicular. If the angle between two vectors is 90°, the cosine of this angle is 0, so we can conclude the following from the angle formula:

(12.2)

$$\boxed{\vec{\mathbf{a}} \perp \vec{\mathbf{b}} \iff \cos\varphi = 0 \implies \cos\varphi \cdot \lvert\vec{\mathbf{a}}\rvert \cdot \lvert\vec{\mathbf{b}}\rvert = 0 \iff \vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = 0}$$



Example 6. Although the vectors $\vec{\mathbf{a}} = (23, 8)$ and $\vec{\mathbf{b}} = (7, -20)$ in this figure may seem to be perpendicular, they are not, because their dot product is

$$\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = 23 \cdot 7 + 8 \cdot (-20) = 1$$

Let's calculate the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

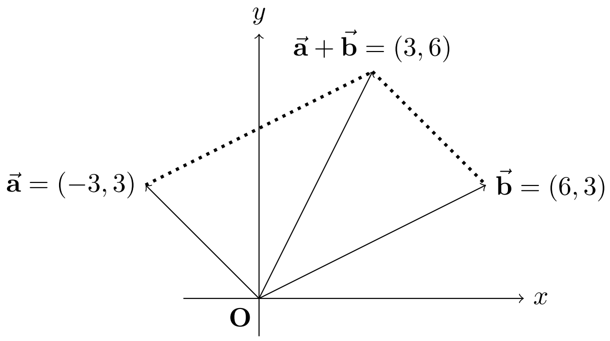
$$\lvert\vec{\mathbf{a}}\rvert = \sqrt{23^2 + 8^2} = \sqrt{593} \ \text{ and } \ \lvert\vec{\mathbf{b}}\rvert = \sqrt{7^2 + (-20)^2} = \sqrt{449}$$

$$\implies \quad \cos\varphi = \frac{(23,8) \bullet (7,-20)}{\lvert(23,8)\rvert \cdot \lvert(7,-20)\rvert} = \frac{1}{\sqrt{593} \cdot \sqrt{449}}$$

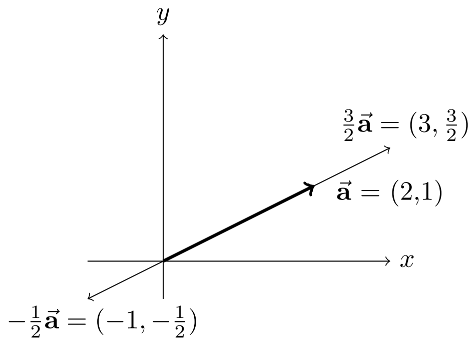
and using a calculator we can find that $\varphi = 89.9$ degrees.

Geometric interpretations of manipulations.

Addition: Geometrically, addition of vectors amounts to drawing a parallelogram:



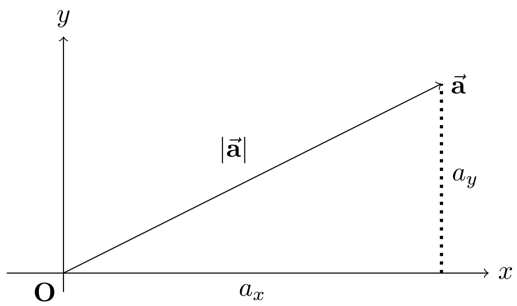
Scalar multiplication: As you can see in the figure below, scalar multiplication does not change the direction of the vector, but only the magnitude:



Magnitude: The geometric interpretation of 'magnitude' is 'length':

(12.3) $|\vec{\mathbf{a}}|$ is the length of the vector $\vec{\mathbf{a}}$

This follows directly from Pythagoras' theorem:



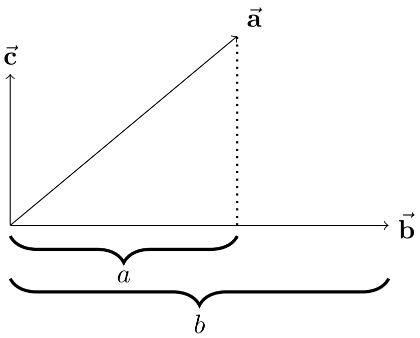
In this figure $\vec{\mathbf{a}} = (a_x, a_y)$. According to Pythagoras the hypotenuse is

$$\sqrt{a_x^2 + a_y^2}$$

which matches our definition of $|\vec{\mathbf{a}}|$.

Dot product: The dot product $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}$ is the product of the length of $\vec{\mathbf{b}}$ and the length of the component of $\vec{\mathbf{a}}$ in the direction of $\vec{\mathbf{b}}$:

$$\boxed{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = ab}$$



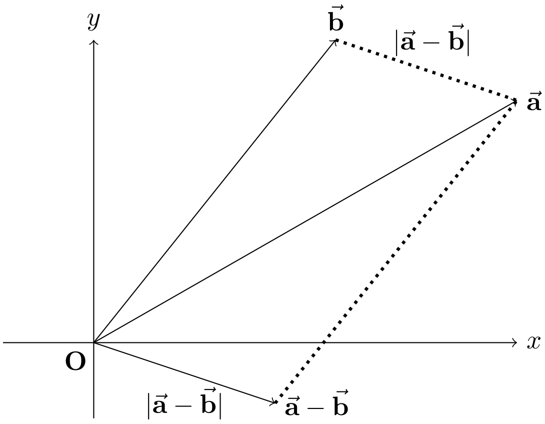
Proof. If α is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, then $a = |\vec{\mathbf{a}}| \cos\alpha$ and hence $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = |\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}| \cdot \cos\alpha = ab$.

Vector $\vec{\mathbf{c}}$ demonstrates the quality that $\vec{\mathbf{c}} \bullet \vec{\mathbf{b}} = 0$ when $\vec{\mathbf{c}}$ is perpendicular to $\vec{\mathbf{b}}$, as the component of $\vec{\mathbf{c}}$ in the direction of $\vec{\mathbf{b}}$ is zero.

You are of course allowed to swap the roles of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in the dot product. Thus, the dot product is the product of the components in the direction of $\vec{\mathbf{a}}$ as well.

Distance between two points. Geometrically, the magnitude of a subtraction of two vectors is the <u>distance</u> between the heads of the vectors:

(12.4) $\quad$ the distance between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $|\vec{\mathbf{a}} - \vec{\mathbf{b}}|$

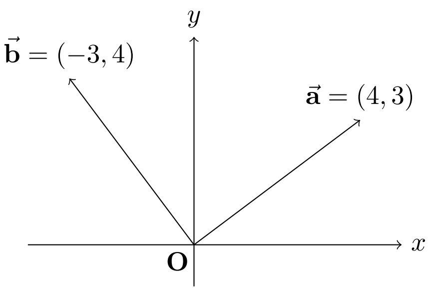


This follows from the fact that in the parallelogram to the right two opposite sides have equal lengths.

If you find it hard to see why $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ is this downwards pointing vector in the figure to the right, it might be helpful to call this vector $\vec{\mathbf{c}}$ for a moment. From the parallelogram, we then can see that

$$\vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{a}} \quad \Longrightarrow \quad \vec{\mathbf{c}} = \vec{\mathbf{a}} - \vec{\mathbf{b}}$$

By the way, if you multiply a vector by the scalar -1, you only change its direction, but its length remains the same. This means that the distance between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is also equal to $|\vec{\mathbf{b}} - \vec{\mathbf{a}}|$.



Example 7. Let $\vec{\mathbf{a}} = (4, 3)$ and $\vec{\mathbf{b}} = (-3, 4)$.

a) Calculate the length of $\vec{\mathbf{a}}$.

b) Calculate the distance from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

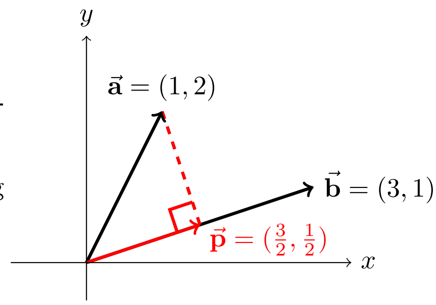
Solution.

a) The length of $\vec{\mathbf{a}}$ is $|\vec{\mathbf{a}}| = |(4, 3)| = \sqrt{4^2 + 3^2} = 5$.

b) The distance from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$ is $|\vec{\mathbf{a}} - \vec{\mathbf{b}}| = |(7, -1)| = \sqrt{50} = 5\sqrt{2}$.

Projection. The <u>projection</u> of a vector $\vec{\mathbf{a}}$ on another vector $\vec{\mathbf{b}}$ is the vector $\vec{\mathbf{p}}$ with the following properties:

- $\vec{\mathbf{p}}$ has the same direction as $\vec{\mathbf{b}}$
- the length of $\vec{\mathbf{p}}$ is such that the line connecting $\vec{\mathbf{a}}$ and $\vec{\mathbf{p}}$ is perpendicular to $\vec{\mathbf{b}}$



You can calculate the projection $\vec{\mathbf{p}}$ of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$ using the following formula:

(12.5) $$\vec{\mathbf{p}} = \frac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{\vec{\mathbf{b}} \bullet \vec{\mathbf{b}}} \cdot \vec{\mathbf{b}}$$

Example 8. The projection of $\vec{\mathbf{a}} = (1, 2)$ onto $\vec{\mathbf{b}} = (3, 1)$ is the red vector $\vec{\mathbf{p}}$ in the figure above:

$$\vec{\mathbf{p}} = \frac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{\vec{\mathbf{b}} \bullet \vec{\mathbf{b}}} \cdot \vec{\mathbf{b}} = \frac{(1, 2) \bullet (3, 1)}{(3, 1) \bullet (3, 1)} \cdot (3, 1) = \frac{1 \cdot 3 + 2 \cdot 1}{3 \cdot 3 + 1 \cdot 1} \cdot (3, 1) = \frac{5}{10}(3, 1) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

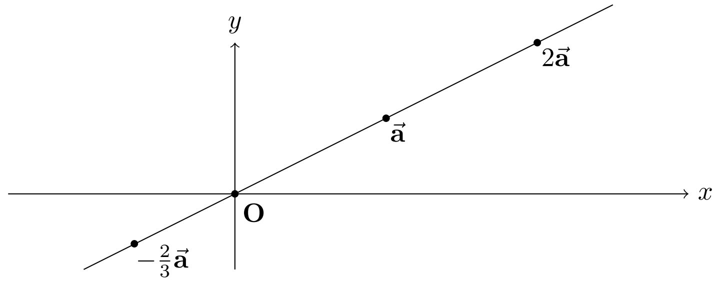
Example 9. Calculate the projection of $\vec{\mathbf{b}} = (3, 1)$ onto $\vec{\mathbf{a}} = (1, 2)$.

Solution. If we call the desired projection vector $\vec{\mathbf{q}}$ this time to distinguish it from $\vec{\mathbf{p}}$ from the previous example, we have

$$\vec{\mathbf{q}} = \frac{\vec{\mathbf{b}} \bullet \vec{\mathbf{a}}}{\vec{\mathbf{a}} \bullet \vec{\mathbf{a}}} \cdot \vec{\mathbf{a}} = \frac{(3, 1) \bullet (1, 2)}{(1, 2) \bullet (1, 2)} \cdot (1, 2) = \frac{3 \cdot 1 + 1 \cdot 2}{1 \cdot 1 + 2 \cdot 2} \cdot (1, 2) = \frac{5}{5}(1, 2) = (1, 2)$$

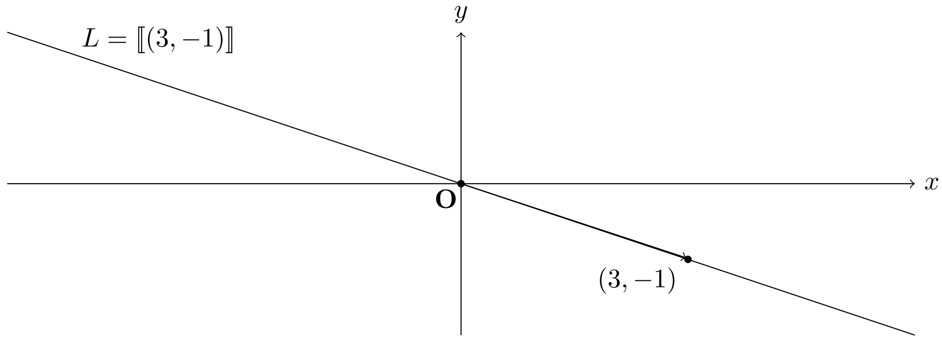
Note that this is different from the projection of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$! Can you see in the figure above why $\vec{\mathbf{q}} = \vec{\mathbf{a}}$?

Lines through the origin. In our geometric interpretation of $\mathbb{R}^2$ the vector $t \cdot \vec{\mathbf{a}}$ lies on the line through the origin and $\vec{\mathbf{a}}$:



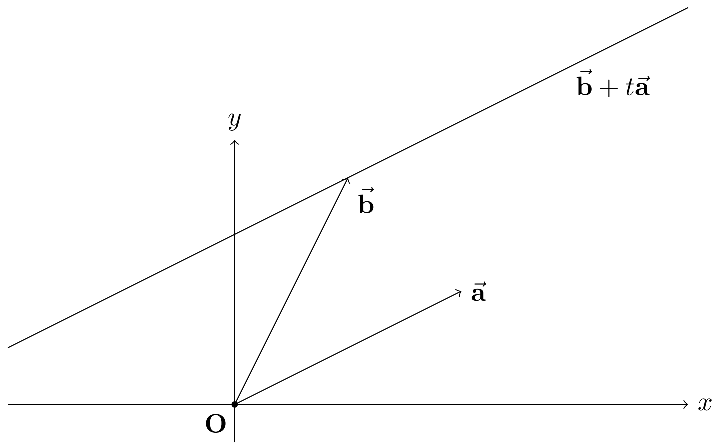
The line through $\mathbf{O}$ and $\vec{\mathbf{a}}$ is the collection of all multiples of $\vec{\mathbf{a}}$, that is all vectors $t \cdot \vec{\mathbf{a}}$ where t is any real number. We denote this line by $[\![\vec{\mathbf{a}}]\!]$ and you can see that this notation is quite similar to $t \cdot \vec{\mathbf{a}}$. This is because the double square brackets $[\![\cdots]\!]$ simply mean 'take any multiple of what is inside these brackets', which is exactly equivalent to $t \cdot \vec{\mathbf{a}}$ with t any real number. The notation $t \cdot \vec{\mathbf{a}}$ is called a <u>parametric representation</u> of the line: if you consider the parameter t to represent time, the line can be viewed as the trajectory that an object takes when it moves with constant velocity, starting at $(0, 0)$ when $t = 0$. The line is then collection of all points over which the object moves.

Example 10. The line $L = [\![(3, -1)]\!]$ is the collection of all multiples of the vector $(3, -1)$:



The <u>equation</u> of this line L is $y = -\frac{1}{3}x$. This means that L is the collection of all vectors (x, y) satisfying $y = -\frac{1}{3}x$.

Other lines in $\mathbb{R}^2$. If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors, then the collection of all vectors $\vec{\mathbf{b}} + t \cdot \vec{\mathbf{a}}$ with t any real number is a line through the point $\vec{\mathbf{b}}$, parallel to the line $[\![\vec{\mathbf{a}}]\!]$:



We call this line $\vec{\mathbf{b}} + [\![\vec{\mathbf{a}}]\!]$ and you can again see that this notation is quite similar to its parametric representation $\vec{\mathbf{b}} + t \cdot \vec{\mathbf{a}}$. We can interpret this notation as follows:

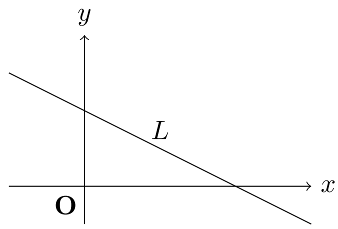
> In order to find all points on the line $\vec{\mathbf{b}} + [\![\vec{\mathbf{a}}]\!]$, you first walk along the vector $\vec{\mathbf{b}}$ and then walk into the direction in which the vector $\vec{\mathbf{a}}$ points (or the opposite direction) for any number of centimetres.

This is why $\vec{\mathbf{a}}$ is often called a <u>direction vector</u> of the line $\vec{\mathbf{b}} + [\![\vec{\mathbf{a}}]\!]$. The vector $\vec{\mathbf{b}}$ is then a <u>supporting vector</u> (in Dutch: "steunvector"), because you could consider this vector to 'support' the line.

Example 11. This line L is given by the equation

$$y = -\frac{1}{2}x + 1$$

Find a parametric representation of L.



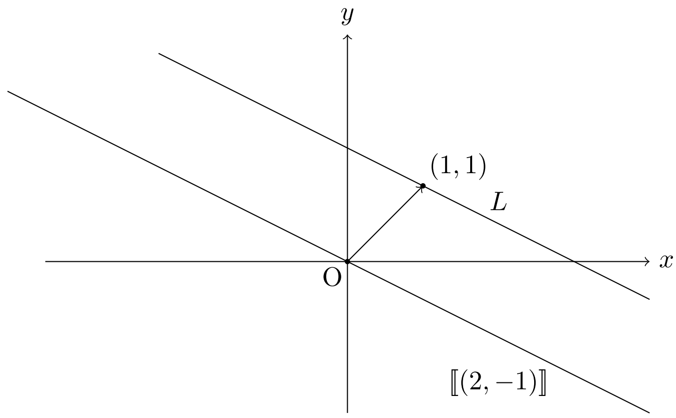
Solution. From the equation $y = -\frac{1}{2}x + 1$ you conclude that a point (x, y) on L satisfies

$$(x, y) = (x, -\tfrac{1}{2}x + 1) = (0, 1) + (x, -\tfrac{1}{2}x) = (0, 1) + x \cdot (1, -\tfrac{1}{2})$$

Now, you've found a parametric representation of L: $(0, 1) + t \cdot (1, -\frac{1}{2}) = (0, 1) + t(2, -1)$.

Example 12. Let L be the line $(1, 1) + [\![(2, -1)]\!]$. Draw L and determine the equation of L.

Solution. L is the line through the point $(1, 1)$ parallel to the line $[\![(2, -1)]\!]$:



In order to find the equation of L you can use the following procedure: a point (x, y) on L can be written as $(1, 1) + t \cdot (2, -1)$. Let's do some algebra:

$$(x, y) = (1, 1) + t \cdot (2, -1) = (1, 1) + (2t, -t) = (1 + 2t, 1 - t) \implies \begin{cases} x = 1 + 2t \\ y = 1 - t \end{cases}$$

From these two equations you can eliminate t: add the first equation ($x = 1 + 2t$) to the second equation multiplied by 2 ($2y = 2 - 2t$). This is how you find the equation of L: $x + 2y = 3$ which is equivalent to $y = -\frac{1}{2}x + \frac{3}{2}$.

Natural basis for $\mathbb{R}^2$. The vectors $(1, 0)$ and $(0, 1)$ form the natural basis for $\mathbb{R}^2$. In physics courses the following notations for these two important vectors are somewhat popular:

$$\boxed{\begin{aligned} \vec{\mathbf{e_x}} &= (1, 0) \\ \vec{\mathbf{e_y}} &= (0, 1) \end{aligned}} \quad \text{or sometimes} \quad \boxed{\begin{aligned} \vec{\mathbf{\imath}} &= (1, 0) \\ \vec{\mathbf{\jmath}} &= (0, 1) \end{aligned}}$$

Every vector from $\mathbb{R}^2$ can be expressed in terms of these basis vectors:

$$(\alpha, \beta) = (\alpha, 0) + (0, \beta) = \alpha(1, 0) + \beta(0, 1) = \alpha\, \vec{\mathbf{e_x}} + \beta\, \vec{\mathbf{e_y}}$$

Example 13.

$$(-3, 7) = (-3, 0) + (0, 7) = -3 \cdot (1, 0) + 7 \cdot (0, 1) = -3\, \vec{\mathbf{e_x}} + 7\, \vec{\mathbf{e_y}} = -3\, \vec{\mathbf{\imath}} + 7\, \vec{\mathbf{\jmath}}$$

Different notations. Some people or books will use different notations for some of the concepts from this chapter. In addition to the column notation for vectors, you might also encounter:

$\langle \vec{\mathbf{a}} \mid \vec{\mathbf{b}} \rangle$ instead of $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}$

$\|\vec{\mathbf{a}}\|$ instead of $|\vec{\mathbf{a}}|$ (which you should not confuse with the line $[\![\vec{\mathbf{a}}]\!]$!)

Exercises chapter 12

Exercise 1. Let $\vec{\mathbf{a}} = (8,3)$ and $\vec{\mathbf{b}} = (-3,6)$.

a) Calculate $\vec{\mathbf{c}} = \vec{\mathbf{a}} + 9\vec{\mathbf{b}}$.

b) Calculate $\vec{\mathbf{p}} = -6\vec{\mathbf{a}} + 4\vec{\mathbf{b}}$.

c) Calculate $\vec{\mathbf{q}} = 7\vec{\mathbf{b}} - 2\vec{\mathbf{a}}$.

d) Calculate $|\vec{\mathbf{r}}|$ if $\vec{\mathbf{r}} = -10\vec{\mathbf{a}} + 3\vec{\mathbf{b}}$.

Exercise 2. Let $\vec{\mathbf{a}} = (5,1)$ and $\vec{\mathbf{b}} = (-2,-4)$.

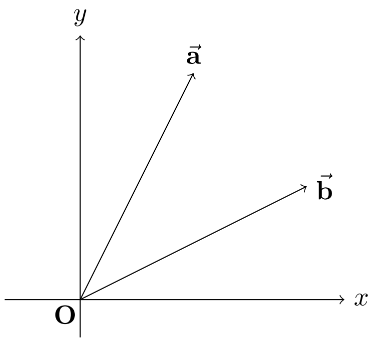
a) Sketch the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

b) Calculate the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

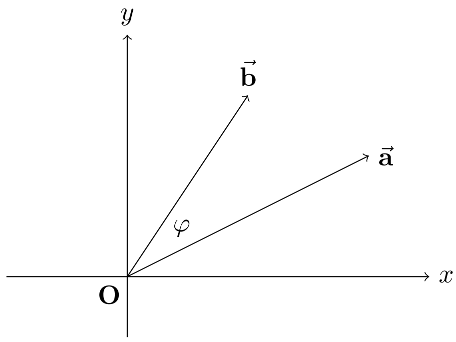
c) Calculate $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ and construct $\vec{\mathbf{c}}$.

d) Calculate $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, $|\vec{\mathbf{c}}|$, and the distance between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Exercise 3. (geometric interpretation of the <u>difference</u> of two vectors)
Construct $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ in the figure below.



Exercise 4. Let $\vec{\mathbf{a}} = (4,2)$ and $\vec{\mathbf{b}} = (2,3)$.



a) Calculate the dot product $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}$.

b) Calculate $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$.

c) Calculate the distance from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

d) Calculate the cosine of the angle φ between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

e) Draw and calculate the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$.

Exercise 5. Express the vector $\vec{\mathbf{a}} = 2\vec{\mathbf{e_x}} - 3\vec{\mathbf{e_y}}$ in the form (a_x, a_y) and construct the line $[\![\vec{\mathbf{a}}]\!]$.

Exercise 6. Draw in $\mathbb{R}^2$ the lines $(2,3) + [\![(1,1)]\!]$ and $(-1,1) + [\![(3,-4)]\!]$ and find their intersection.

Exercise 7. Find a parametric representation of the line through $(1,2)$ and $(5,0)$.

Exercise 8. Find the equation of the line with parametric representation $(1,3) + t \cdot (5,2)$.

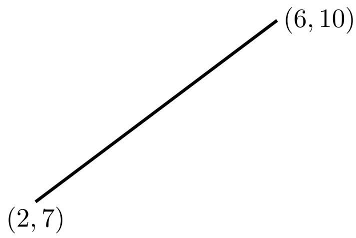
Exercise 9. Find a parametric representation of the line with equation $y = 7 - 3x$.

Exercise 10. (units: metres, seconds)
A little red pussycat runs with constant speed in precisely one second from the starting point $(2, 7)$ to the end point $(6, 10)$.

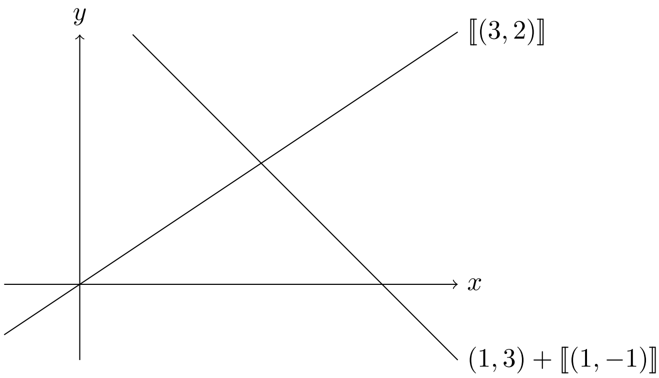
a) What is its speed?

b) Where is the pussycat after t seconds ($0 \le t \le 1$)?

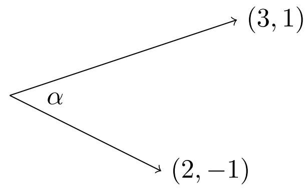
c) What is the pussycat's velocity vector?



Exercise 11. Calculate the intersection point of the line $[\![(3, 2)]\!]$ with the line $(1, 3) + [\![(1, -1)]\!]$.



Exercise 12. Calculate the angle α between the vectors $(3, 1)$ and $(2, -1)$.



Exercise 13. Prove the following statement: if two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are perpendicular, then the projection of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$ is the zero vector $(0, 0)$.

Exercise 14. Which point on the line $[\![(2, -1)]\!]$ is closest to the point $(5, 0)$?

CHAPTER 13

Solutions to all exercises

Exercise 1.1.

a) $-2 \in \mathbb{N}$ is false: the natural numbers are all positive whole numbers (and 0), but -2 is negative.

b) $\frac{1}{3} \in \mathbb{R}$ is true, because fractions are part of the set of all real numbers.

c) $5 \in \mathbb{Q}$ is true, because you can write the number 5 as a fraction in many different ways, for example $\frac{5}{1}$.

d) $\sqrt{2} \in \mathbb{Q}$ is false, because roots are not rational. There exists a nice 'proof by contradiction' for this statement, but you will need to know some theory from the next chapters to appreciate it, so I will not bother you with it here (but don't hesitate to Google it if you are interested).

e) $\pi \in \mathbb{C}$ is true, because π is a real number and the set of all real numbers is a subset of the set of all complex numbers.

Exercise 1.2. I found $283 + 1729 = 2012$, which I did as follows:

$$\boxed{\begin{array}{rr} & 283 \\ + & 1729 \\ \hline & 2012 \end{array}}$$

Exercise 1.3. Hopefully someone told you in the past that $\cdot$ has priority over $+$ and $-$, so what I asked you was to calculate $(635 \cdot 728) - (208 \cdot 728)$. You can use the multiplication rule $x(y - z) = xy - xz$ from right to left to simplify this: we have $x = 728$, $y = 635$ and $z = 208$, so $635 \cdot 728 - 208 \cdot 728$ is equal to $(635 - 208) \cdot 728$. The actual calculation requires two steps:

Step 1: $\boxed{\begin{array}{rr} & 635 \\ - & 208 \\ \hline & 427 \end{array}}$ $\implies$ Step 2: $\boxed{\begin{array}{rr} & 427 \\ \times & 728 \\ \hline & 3416 \\ & 854 \\ + & 2989 \\ \hline & 310856 \end{array}}$

Exercise 1.4. I use the golden rule of algebra:

$$673 + 4x = 841 \xRightarrow{\text{subtract 673 from both sides}} 4x = 841 - 673 \xRightarrow{\text{divide both sides by 4}} x = \frac{841 - 673}{4}$$

I did the actual calculation as follows: first $\boxed{\begin{array}{rr} & 841 \\ - & 673 \\ \hline & 168 \end{array}}$ and then $\boxed{\begin{array}{l} 4 \,/\, 168 \,\backslash\, 42 \\ \quad -\underline{16} \\ \qquad 08 \\ \quad\; -\underline{\;\,8} \\ \qquad\;\; 0 \end{array}}$. Hence, $\boxed{x = 42}$.

Exercise 1.5. This is a job for the golden rule of algebra:

$$\frac{2\alpha - 3}{\alpha + 1} = 3 \xRightarrow{\text{multiply by } \alpha+1} 2\alpha - 3 = 3(\alpha + 1) = 3\alpha + 3 \xRightarrow{\text{subtract } 2\alpha} -3 = \alpha + 3 \xRightarrow{\text{subtract 3}} \boxed{\alpha = -6}$$

Exercise 1.6. Let's use the golden rule of algebra again:

$$\frac{2}{p} + 1 = \frac{3}{2p} + 1 \xRightarrow{\text{multiply by } 2p} 4 + 2p = 3 + 2p \xRightarrow{\text{subtract } 2p} 4 = 3 \implies ???$$

4 is of course never equal to 3, so your conclusion should be that this equation has $\boxed{\text{no solutions}}$.

Exercise 2.1. I evaluate both $y(x+2)$ and $(x+3)(y-1)$ separately and then bring all the y's to one side:

- $y(x+2) = yx + 2y$
- $(x+3)(y-1) = x(y-1) + 3(y-1) = xy - x + 3y - 3$
- so $y(x+2) = (x+3)(y-1) \iff yx + 2y = xy - x + 3y - 3 \iff 2y - 3y = -x - 3 \iff y = x + 3$

Exercise 2.2. I can't give you a solution to learn the notable products by heart, but I can prove them for you by expanding the brackets:

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) = x(x+y) + y(x+y) = (x^2+xy) + (yx+y^2) = x^2 + 2xy + y^2 \\
(x-y)^2 &= (x-y)(x-y) = x(x-y) - y(x-y) = (x^2-xy) - (yx-y^2) = x^2 - 2xy + y^2 \\
(x+y)(x-y) &= x(x-y) + y(x-y) = (x^2-xy) + (yx-y^2) = x^2 - y^2
\end{aligned}$$

Exercise 2.3.

a) These are all incorrect. You can prove this by replacing x, y and z by the number 1: the results won't make any sense.

b)

$$\boxed{\begin{aligned}
x^3 + x^4 &= x^3(1+x) \\
(x+y+z)^2 &= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \\
x - (y-z) &= x - y + z \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}}$$

For the last one, you can write $(x+y)^3$ as $(x+y)(x+y)^2$, which is equal to $(x+y)(x^2+2xy+y^2)$. If you now expand the brackets, you should find the same result as I did.

Exercise 2.4. $5^3 + 5^4 = 5^3(1+5) = 125 \cdot 6 = 750$

Exercise 2.5. $(a+3)^2 - (a-3)^2 = (a^2+6a+9) - (a^2-6a+9) = a^2 + 6a + 9 - a^2 + 6a - 9 = 12a$

Exercise 2.6. $\dfrac{9^{27}}{3^{50}} = \dfrac{(3^2)^{27}}{3^{50}} = \dfrac{3^{54}}{3^{50}} = 3^4 = 81$

Exercise 2.7. Use one of the notable products: $\dfrac{a^2-b^2}{a+b} = \dfrac{(a+b)(a-b)}{a+b} = a - b$

Exercise 2.8. You have the pleasant choice between hard calculation:

$$\begin{array}{r}
378 \\
\times\ 378 \\ \hline
3024 \\
2646 \\
+\ 1134 \\ \hline
142884
\end{array}
\qquad
\begin{array}{l}
377\ /\ 142883\ \backslash\ 379 \\
\quad -\ 1131 \\ \hline
\qquad\ \ 2978 \\
\quad -\ 2639 \\ \hline
\qquad\quad\ 3393 \\
\quad\ \ -\ 3393 \\ \hline
\qquad\qquad\ \ 0
\end{array}$$

and cleverly using the previous exercise with $a = 378$ and $b = -1$:

$$\frac{378^2 - 1}{377} = \frac{378^2 - (-1)^2}{378 - 1} = \frac{(378+1)(378-1)}{378-1} = 378 + 1 = 379$$

Exercise 2.9. These are the numbers 0, 5 and −5. One way to find these is as follows:

$$\begin{aligned} x^3 = 25x &\iff x^3 - 25x = 0 \\ &\iff x(x^2-25) = 0 \\ &\iff x(x+5)(x-5) = 0 \\ &\iff x = 0 \;\text{ or }\; x+5 = 0 \;\text{ or }\; x-5 = 0 \\ &\iff x = 0 \;\text{ or }\; x = -5 \quad \text{or }\; x = 5 \end{aligned}$$

Exercise 2.10. $2^{30} = (2^3)^{10} = 8^{10}$ and $3^{20} = (3^2)^{10} = 9^{10}$, so 3^{20} is greater than 2^{30}

Exercise 2.11. I use the expression derived in exercise 3, where for y I consecutively take 1 and -1

$$(x+1)^3 - (x-1)^3 = (x^3 + 3x^2 + 3x + 1) - (x^3 - 3x^2 + 3x - 1) = 6x^2 + 2$$

Exercise 3.1. Using long division you can find the following periodic decimal continuation:

$$\frac{1}{7} = 0.\boxed{142857}\boxed{142857}\boxed{142857}\boxed{142857}\boxed{142857}\boxed{142857}\boxed{142857}\cdots\cdots$$

Exercise 3.2.

a) Fractions with the same denominator can simply be added or subtracted: $\dfrac{34}{22} + \dfrac{12}{22} = \dfrac{46}{22} = \dfrac{23}{11}$

b) Fractions with the same numerator **cannot** simply be added or subtracted! You should first make the denominators equal:

$$\frac{5}{3} - \frac{5}{7} = \frac{35}{21} - \frac{15}{21} = \frac{20}{21}$$

c) $\dfrac{41}{75} - \dfrac{9}{20} = \dfrac{41 \cdot 4}{75 \cdot 4} - \dfrac{9 \cdot 15}{20 \cdot 15} = \dfrac{164}{300} - \dfrac{135}{300} = \dfrac{164 - 135}{300} = \dfrac{29}{300}$

Exercise 3.3. Try to find common factors in both the numerator and denominator, and continue until you cannot simplify further:

$$\frac{117}{819} = \frac{3 \cdot 39}{3 \cdot 273} = \frac{39}{273} = \frac{3 \cdot 13}{3 \cdot 91} = \frac{13}{91} = \frac{13}{7 \cdot 13} = \frac{1}{7}$$

If you are secretly using a computer to help you with your calculations, you can also command it to calculate the greatest common divisor of 117 and 819 for you, which happens to be 117. This means that you can just divide both 117 and 819 by 117 to simplify this fraction.

Exercise 3.4. First, make the denominators equal:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

You can now add the two fractions to obtain the desired (but terrible) calculation rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Exercise 3.5. $\dfrac{7}{25}x \cdot \dfrac{3}{14} = \dfrac{9}{100} \iff \dfrac{3}{50}x = \dfrac{9}{100} \iff x = \dfrac{450}{300} = \dfrac{3}{2}$

Exercise 3.6. I use cross-multiplication:

$$\begin{aligned} \frac{x+3}{x+7} = \frac{x-1}{x+1} \quad &\iff \quad (x+3)(x+1) = (x+7)(x-1) \\ &\iff \quad x^2 + 4x + 3 = x^2 + 6x - 7 \quad \iff \quad -2x = -10 \quad \iff \quad x = 5 \end{aligned}$$

Exercise 3.7. Let's start the calculation by substituting the values for x and y:

$$\frac{x+y}{x-y} = \frac{\frac{7}{2}+\frac{14}{5}}{\frac{7}{2}-\frac{14}{5}}$$

Now multiply the numerator and denominator by $2 \cdot 5$:

$$\cdots = \frac{7\cdot 5 + 14 \cdot 2}{7 \cdot 5 - 14 \cdot 2} = \frac{35+28}{35-28} = \frac{63}{7} = 9$$

Exercise 3.8. I hope you're familiar with the concept of 'velocity':

$$\boxed{\text{velocity} = \frac{\text{distance}}{\text{time}}} \qquad \boxed{\text{time} = \frac{\text{distance}}{\text{velocity}}}$$

The first part of the route takes me $\frac{5}{5}$ hours, the second part takes $\frac{5}{4}$ hours, and the third part takes $\frac{5}{3}$ hours. The duration of the total walk adds up to

$$\frac{5}{5}+\frac{5}{4}+\frac{5}{3} = \frac{60}{60}+\frac{75}{60}+\frac{100}{60} = \frac{60+75+100}{60} = \frac{235}{60} \text{ hours}$$

which means that my average velocity is

$$\frac{15}{\frac{235}{60}} = \frac{15\cdot 60}{235} = \frac{3 \cdot 60}{47} = \frac{180}{47} \text{ km/h}$$

This is less than 4 km/h, so if you naively believed that averaging the three velocities would work just as well, then you have to take a careful second look at this exercise.

Exercise 3.9.

$$\beta = \frac{3\alpha+2}{5\alpha-1} \implies \beta(5\alpha-1) = 3\alpha+2 \implies 5\alpha\beta - \beta = 3\alpha+2$$
$$\implies 5\alpha\beta - 3\alpha = \beta + 2 \implies \alpha(5\beta-3) = \beta+2 \implies \alpha = \frac{\beta+2}{5\beta-3}$$

Exercise 3.10. This is done as follows:

$$C = \frac{5}{9}(F-32) \implies \frac{9}{5}C = F - 32 \implies F = 32 + \frac{9}{5}C$$

Exercise 3.11. Yes, because

$$(-0.5)^{-2} = \frac{1}{(-0.5)^2} = \frac{1}{0.25} = 4 \qquad -0.5^{-2} = -\frac{1}{0.5^2} = -\frac{1}{0.25} = -4 \qquad 0.5^2 = 0.25$$

Exercise 3.12. $\dfrac{x^{-3}}{(3x^2)^{-2}} = \dfrac{(3x^2)^2}{x^3} = \dfrac{9x^4}{x^3} = 9x$

Exercise 3.13. The way to go here is to write everything in terms of the denominator $1-t^2$, because we can conveniently use the notable product $(1-t)(1+t) = 1-t^2$:

$$x = \frac{1}{1-t}+\frac{1}{1+t}+\frac{2}{t^2-1} = \frac{1+t}{1-t^2}+\frac{1-t}{1-t^2}+\frac{-2}{1-t^2} = \frac{1+t+1-t-2}{1-t^2} = \frac{0}{1-t^2} = 0$$

Conclusion: I'm not exactly rich.

Exercise 3.14. This is a job for the golden rule of algebra together with all newly acquired calculation rules. I will just give you the final answers:

a) $x = \frac{11}{24}$ b) $x = 6$ c) $x = 56$

Exercise 3.15.

a) Dividing by a fraction is equal to multiplication by the inverse:

$$\frac{\frac{3-3x}{4x-2}}{\frac{x-1}{1-2x}} = \frac{3-3x}{4x-2} \cdot \frac{1-2x}{x-1} = \frac{(3-3x)(1-2x)}{(4x-2)(x-1)} = \frac{3-9x+6x^2}{4x^2-6x+2} = \frac{3(2x^2-3x+1)}{2(2x^2-3x+1)} = \frac{3}{2}$$

However, this fraction is only equal to $\frac{3}{2}$ for those values of x that do not yield a zero denominator, so $x \neq 1$ and $x \neq \frac{1}{2}$.

b) The denominator can be factored into $(x-4)(x+1)$, so let's try if partial fraction decomposition yields a simpler expression:

$$\frac{\alpha}{x-4} + \frac{\beta}{x+1} = \frac{\alpha(x+1)+\beta(x-4)}{(x-4)(x+1)} = \frac{(\alpha+\beta)x + (\alpha - 4\beta)}{x^2-3x-4}$$

For the numerator to be equal to the original fraction, we have $\alpha+\beta = 0$ (because the original fraction doesn't have a power of x in the numerator) and $\alpha - 4\beta = 5$. The first of these equations yields $\beta = -\alpha$ and substitution into the second gives

$$\alpha - 4\beta = \alpha - 4(-\alpha) = 5\alpha = 5 \quad \Longrightarrow \quad \alpha = 1 \quad \Longrightarrow \quad \beta = -1 \quad \Longrightarrow \quad \boxed{\frac{5}{x^2-3x-4} = \frac{1}{x-4} - \frac{1}{x+1}}$$

c) The degree of the numerator is greater than the degree of the denominator, so we can use long division:

$$\begin{array}{rrrrrrrrl}
2x+3 & / & 8x^2 & - & 2x & + & 1 & \backslash & 4x - 7 + \frac{22}{2x+3} \\
 & - & 8x^2 & + & 12x & & & & \\ \hline
 & & & & -14x & + & 1 & & \\
 & & & - & -14x & - & 21 & & \\ \hline
 & & & & & & 22 & &
\end{array}$$

$2x$ fits $4x$ times into $8x^2$, so we subtract $4x$ times $2x+3$. We are left with $-14x+1$, so we subtract -7 times $2x+3$. The remainder 22 cannot be divided by $2x+3$ anymore.

Conclusion: $\dfrac{8x^2-2x+1}{2x+3} = 4x - 7 + \dfrac{22}{2x+3}$

d) This is a job for partial fraction decomposition, because $x^2+3x+2 = (x+1)(x+2)$:

$$\frac{\alpha}{x+1} + \frac{\beta}{x+2} = \frac{\alpha(x+2)+\beta(x+1)}{(x+1)(x+2)} = \frac{(\alpha+\beta)x + (2\alpha+\beta)}{x^2+3x+2}$$

The original fraction's numerator has -1 as coefficient of x and 1 as additional constant, so we have $\alpha+\beta = -1$, and $2\alpha+\beta = 1$. The first of these equations yields $\beta = -1-\alpha$ and substitution into the second gives

$$2\alpha + \beta = 2\alpha + (-1-\alpha) = \alpha - 1 = 1 \quad \Longrightarrow \quad \alpha = 2$$

Now, we have

$$\beta = -1 - \alpha = -1 - 2 = -3 \quad \Longrightarrow \quad \boxed{\frac{1-x}{x^2+3x+2} = \frac{2}{x+1} - \frac{3}{x+2}}$$

e) The degree of $2x^2-x-6$ (which is 2) is greater than the degree of $x-2$ (which is 1), so use long division:

$$\begin{array}{rrrrrrrrl}
x-2 & / & 2x^2 & - & x & - & 6 & \backslash & 2x+3 \\
 & - & 2x^2 & - & 4x & & & & \\ \hline
 & & & & 3x & - & 6 & & \\
 & & & - & 3x & - & 6 & & \\ \hline
 & & & & & & 0 & &
\end{array}$$

x fits $2x$ times into $2x^2$, so we subtract $2x$ times $x-2$. We are left with $3x-6$, which fits exactly 3 times $x-2$.

Conclusion: $\dfrac{2x^2-x-6}{x-2} = 2x+3$, but we should not forget that the simplified expression $2x+3$ is only equal to our original fraction as long as its denominator $x-2$ is not equal to zero, so $x \neq 2$.

Exercise 4.1. For negative x this 'rule' is incorrect. For example:

$$\sqrt{(-5)^2} = \sqrt{(-5)\cdot(-5)} = \sqrt{25} = 5 \quad \text{(and not } -5\text{)}$$

Exercise 4.2. Their squares are

$$x^2 = \frac{3}{5} \qquad\qquad y^2 = \frac{8}{13}$$

and I can compare those with ease when I write them such that they have equal denominators:

$$x^2 = \frac{39}{65} \qquad\qquad y^2 = \frac{40}{65}$$

Now it's obvious that $x < y$.

Exercise 4.3. If two numbers are equal, then their squares are also equal:

$$\sqrt{x+3} = 2\sqrt{x-1} \implies \left(\sqrt{x+3}\right)^2 = \left(2\sqrt{x-1}\right)^2$$

$$\implies x+3 = 4(x-1) \implies x+3 = 4x-4 \implies 7 = 3x \implies x = \frac{7}{3}$$

Exercise 4.4. $x = 49$, because:

$$\sqrt{x+15} + \sqrt{x} = 15 \iff (\sqrt{x+15})^2 = (15-\sqrt{x})^2 \iff x+15 = 225 - 30\sqrt{x} + x \iff 30\sqrt{x} = 210$$

Exercise 4.5.

a) $\sqrt{12} - \sqrt{3} = \sqrt{4\cdot 3} - \sqrt{3} = 2\sqrt{3} - \sqrt{3} = \sqrt{3}$

b) $\dfrac{13}{\sqrt{5}} - \sqrt{20} = \dfrac{13}{5}\sqrt{5} - 2\sqrt{5} = \left(\dfrac{13}{5} - 2\right)\sqrt{5} = \dfrac{3}{5}\sqrt{5}$

c) For something like this I use the square root trick:

$$\frac{3+5\sqrt{2}}{1+\sqrt{2}} = \frac{(3+5\sqrt{2})(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{3-3\sqrt{2}+5\sqrt{2}-10}{1-2} = \frac{2\sqrt{2}-7}{-1} = 7-2\sqrt{2}$$

d) I use the formula $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$:

$$\left(2+\sqrt{3}\right)^3 = 8 + 12\sqrt{3} + 18 + 3\sqrt{3} = 26 + 15\sqrt{3}$$

Exercise 4.6. $|x+321| = |x-750|$ means: the distance of x to -321 is equal to the distance of x to 750. In other words: x lies exactly halfway between -321 and 750, so

$$x = \frac{-321+750}{2} = \frac{429}{2}$$

Exercise 4.7.

a) $\dfrac{9-x}{3+\sqrt{x}} = \dfrac{(3+\sqrt{x})(3-\sqrt{x})}{3+\sqrt{x}} = 3-\sqrt{x}$

b) $\dfrac{x\sqrt{6}}{\sqrt{3x}} = \dfrac{x\sqrt{6}}{\sqrt{3}\cdot\sqrt{x}} = \dfrac{x}{\sqrt{x}}\cdot\dfrac{\sqrt{6}}{\sqrt{3}} = \sqrt{x}\cdot\sqrt{2} = \sqrt{2x}$

c) I divide nominator and denominator by $\sqrt{x}$ and then I perform my square root trick:

$$\frac{\sqrt{3x}-\sqrt{x}}{\sqrt{3x}+\sqrt{x}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

d) $\dfrac{x-\sqrt{x}}{1-\sqrt{x}} = \dfrac{\sqrt{x}\cdot(\sqrt{x}-1)}{1-\sqrt{x}} = -\sqrt{x}$

Exercise 4.8. A solution x has to fulfil $3x = \pm(x-1)$. I evaluate both possibilities:

$$\begin{aligned} 3x = x - 1 \quad &\Longrightarrow \quad 2x = -1 \quad \Longrightarrow \quad x = -\tfrac{1}{2} \\ 3x = 1 - x \quad &\Longrightarrow \quad 4x = 1 \quad\;\;\, \Longrightarrow \quad x = \tfrac{1}{4} \end{aligned}$$

The first one is impossible, because when I substitute $x = -\frac{1}{2}$ in the given equation it says $-\frac{3}{2} = \frac{3}{2}$ and this is not correct. On the other hand, the other option seems to be possible after we substitute: $x = \frac{1}{4}$

Exercise 4.9. I square both sides of this equation:

$$\sqrt{1+x^2} = 7 + x \quad \Longrightarrow \quad 1 + x^2 = (7+x)^2 = 49 + 14x + x^2 \quad \Longrightarrow \quad 14x = -48 \quad \Longrightarrow \quad x = -\frac{48}{14} = -\frac{24}{7}$$

Exercise 4.10.

a) $25^{-\frac{1}{2}} = \dfrac{1}{25^{\frac{1}{2}}} = \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$

b) $8^{\frac{2}{3}} = 8^{\frac{1}{3}\cdot 2} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

c) $2^{\frac{7}{2}}/4 = \dfrac{2^{\frac{7}{2}}}{2^2} = 2^{\frac{7}{2}-2} = 2^{\frac{3}{2}} = 2^{1+\frac{1}{2}} = 2^1 \cdot 2^{\frac{1}{2}} = 2\sqrt{2}$

Exercise 4.11.

a) $2^{x-2} = \left(\dfrac{1}{2}\right)^{2x} \quad \Longrightarrow \quad 2^{x-2} = 2^{-2x} \quad \Longrightarrow \quad x - 2 = -2x \quad \Longrightarrow \quad 3x = 2 \quad \Longrightarrow \quad x = \dfrac{2}{3}$

b) $8^{4x} = 32 \quad \Longrightarrow \quad \left(2^3\right)^{4x} = 2^5 \quad \Longrightarrow \quad 2^{12x} = 2^5 \quad \Longrightarrow \quad 12x = 5 \quad \Longrightarrow \quad x = \dfrac{5}{12}$

c) $\left(\dfrac{1}{5}\right)^{2x+5} = 125^x \quad \Longrightarrow \quad 5^{-(2x+5)} = 5^{3x} \quad \Longrightarrow \quad -(2x+5) = 3x \quad \Longrightarrow \quad -5 = 5x \quad \Longrightarrow \quad x = -1$

Exercise 4.12. $\sqrt[9]{7^n} = \sqrt[n]{7^4} \quad \Longrightarrow \quad 7^{\frac{n}{9}} = 7^{\frac{4}{n}} \quad \Longrightarrow \quad \dfrac{n}{9} = \dfrac{4}{n} \quad \Longrightarrow \quad n^2 = 36 \quad \Longrightarrow \quad n = \pm 6$
So the requested natural number n is 6.

Exercise 4.13. The old calculation rules for exponents are indeed valid for the new a^x, but I attached the condition $a > 0$ to these. That condition is essential with the calculation rule $(x^n)^m = x^{nm}$, and Sally indeed tried to fool you with her step

$$(-7)^{2\cdot\frac{1}{2}} = \left((-7)^2\right)^{\frac{1}{2}}$$

Exercise 4.14.

a) Dividing on both sides by 3 yields $x^2 = \frac{4}{3}$. This equation has two solutions: $x = \sqrt{\frac{4}{3}}$ and $x = -\sqrt{\frac{4}{3}}$, which we can simplify to $x = \frac{2}{3}\sqrt{3}$ and $x = -\frac{2}{3}\sqrt{3}$.

b) $x^2 - x - 20 = 0$ can be factored: $(x-5)(x+4) = 0$, which has two solutions: $x = 5$ and $x = -4$.

c) We can solve $x^2 - 2x - 2 = 0$ using the quadratic formula or completing the square (see Chapter 8) to obtain $x = 1 + \sqrt{3}$ or $x = 1 - \sqrt{3}$.

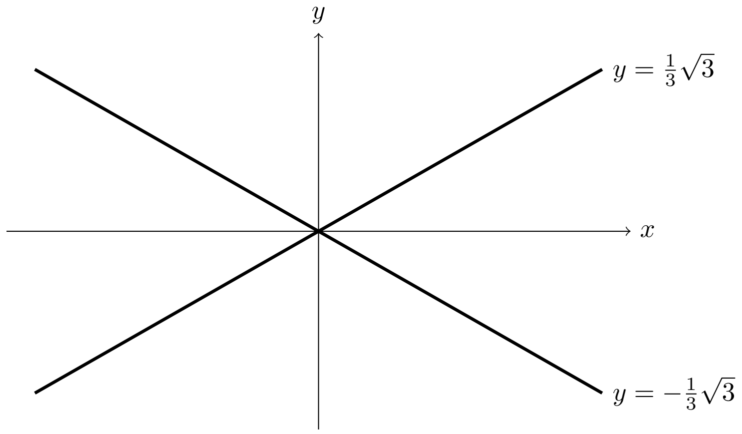
d) Transforming 3 to the left-hand side and dividing on both sides by 2 yields $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$. This equation can be solved using Chapter 8 techniques, but if you're very clever, you can also factor it:

$$x^2 + \tfrac{5}{2}x - \tfrac{3}{2} = 0 \quad \Longleftrightarrow \quad \left(x - \tfrac{1}{2}\right)(x+3) = 0 \quad \Longleftrightarrow \quad x = \tfrac{1}{2} \text{ or } x = -3$$

Exercise 5.1. If we take the square root of both sides of the equation $x^2 = 3y^2$, we obtain

$$x = \pm y\sqrt{3} \quad \Longleftrightarrow \quad \frac{x}{\sqrt{3}} = \pm y \quad \Longleftrightarrow \quad y = \pm\frac{x}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \cdot x = \pm\frac{1}{3}\sqrt{3} \cdot x$$

Conclusion: the curve consists of the lines $y = \frac{1}{3}\sqrt{3} \cdot x$ and $y = -\frac{1}{3}\sqrt{3} \cdot x$:



Exercise 5.2. The slope of this line equals $\frac{1}{3}$, because if you walk from $(-1, 0)$ to $(2, 1)$, then your y-increase is $\frac{1}{3}$ times as big as your x-increase. Therefore, the line is of the type

$$y = \frac{1}{3}x + b$$

Now you calculate b by (for example) substituting the point $(-1, 0)$:

$$0 = \frac{1}{3} \cdot (-1) + b \quad \Longrightarrow \quad b = \frac{1}{3}$$

So the equation of the line is $y = \frac{1}{3}x + \frac{1}{3}$, which you can slightly improve aesthetically by multiplying by 3 on both sides: $3y = x + 1$

Exercise 5.3. Let's write the equation in the form $y = ax + b$:

$$2x + 3y + 1 = 0 \quad \Longrightarrow \quad 3y = -2x - 1 \quad \Longrightarrow \quad y = -\frac{2}{3}x - \frac{1}{3}$$

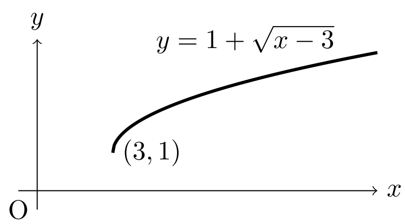
Conclusion: the slope is $-\frac{2}{3}$.

Exercise 5.4. The intersection point (x, y) has to satisfy both equations, so

$$\left.\begin{array}{lcl} x + 2y = 1 & \Longrightarrow & x = 1 - 2y \\ 3y = x - 2 & \Longrightarrow & x = 3y + 2 \end{array}\right\} \quad \Longrightarrow \quad 1 - 2y = 3y + 2 \quad \Longrightarrow \quad 5y = -1 \quad \Longrightarrow \quad y = -\frac{1}{5}$$

and the rest is simple: by substituting $y = -\frac{1}{5}$ in one of the equations, you find $x = \frac{7}{5}$. The intersection point is therefore $(\frac{7}{5}, -\frac{1}{5})$.

Exercise 5.5.



The domain of the function $y = 1 + \sqrt{x - 3}$ is $[3, \infty)$. This is interval notation for the collection of all real numbers ≥ 3. For numbers < 3 the root is negative, which is impossible. The range is $[1, \infty)$, because the minimum value of the root is 0, so y will never reach values below 1.

Exercise 5.6. The intersection point (x, y) satisfies

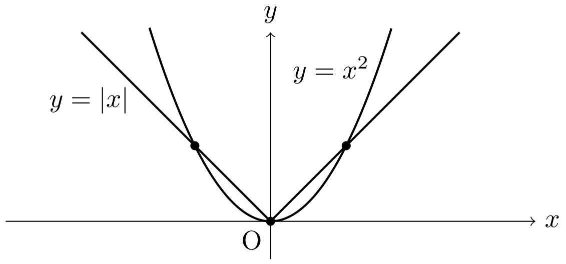
$$\left.\begin{array}{l} y = \sqrt[3]{1 - 2x} \\ y = 2 \end{array}\right\} \quad \Longrightarrow \quad \sqrt[3]{1 - 2x} = 2 \quad \Longrightarrow \quad 1 - 2x = 8 \quad \Longrightarrow \quad 2x = -7 \quad \Longrightarrow \quad x = -\frac{7}{2}$$

Conclusion: the intersection point is $(-\frac{7}{2}, 2)$.

Exercise 5.7. Direct calculations involving absolute values can be complicated, which is why I split up my calculations into two cases:

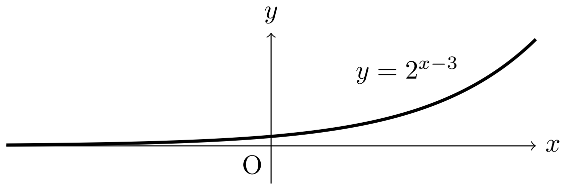
Case 1: $x \geq 0$. An intersection point now satisfies $x^2 = x$, so $x^2 - x = 0$, so $x(x-1) = 0$, so $x = 0$ or $x = 1$.

Case 2: $x < 0$. An intersection point now satisfies $x^2 = -x$ and therefore (divide by x) $x = -1$.



A quick check with our original equation $x^2 = |x|$ shows that $x = -1$, $x = 0$ and $x = 1$ are indeed solutions, so we found three intersection points: $(-1, 1)$, $(0, 0)$ and $(1, 1)$.

Exercise 5.8.



$$2^{x-3} = 64 = 2^6 \quad \Longrightarrow \quad x - 3 = 6 \quad \Longrightarrow \quad x = 9$$

Exercise 5.9. I am going to express y in terms of x:

$$y = x^2 - yx^2 \quad \Longrightarrow \quad y + yx^2 = x^2 \quad \Longrightarrow \quad y(1+x^2) = x^2 \quad \Longrightarrow \quad \boxed{y = \frac{x^2}{1+x^2}}$$

Exercise 5.10.

a) $y = \dfrac{3-x}{4} \quad \Longrightarrow \quad 4y = 3 - x \quad \Longrightarrow \quad x = 3 - 4y \quad \Longrightarrow \quad \boxed{p(x) = 3 - 4x}$ is the inverse of f.

b) Let's first rewrite g:

$$g(x) = x^2 + 2x + 1 = (x+1)^2 \quad \Longrightarrow \quad y = (x+1)^2 \quad \Longrightarrow \quad x + 1 = +\sqrt{y} \text{ or } x + 1 = -\sqrt{y}$$

Let's check whether the $+$ or $-$ sign is correct by plugging in any value of x in the original function rule: if we take for example $x = 1$, then $g(1) = 4$, so plugging in $y = 4$ should return $x = 1$. This happens with the $+$ sign but not with the $-$ sign, so we need to continue with the $+$ sign:

$$x + 1 = \sqrt{y} \quad \Longrightarrow \quad x = \sqrt{y} - 1 \quad \Longrightarrow \quad \boxed{q(x) = \sqrt{x} - 1} \text{ is the inverse of } g \text{ on the domain } x \geq 0$$

c) $y = \sqrt[3]{2-x} \quad \Longrightarrow \quad y^3 = 2 - x \quad \Longrightarrow \quad x = 2 - y^3 \quad \Longrightarrow \quad \boxed{r(x) = 2 - x^3}$ is the inverse of h.

Exercise 5.11. The distance between $(-6, 3)$ and $(5, 1)$ is

$$\sqrt{(5-(-6))^2 + (1-3)^2} = \sqrt{121 + 4} = \sqrt{125} = 5\sqrt{5}$$

Exercise 5.12. The radius of this circle is the distance from $(2, 1)$ to the origin, which is $\sqrt{2^2 + 1^2} = \sqrt{5}$. Its area is therefore 5π.

Exercise 5.13.

a) The radius is 3 (because this is $\frac{1}{2\pi}$ times the circumference), so the equation of the circle is

$$(x-3)^2 + (y-0)^2 = 9 \quad \Longrightarrow \quad x^2 - 6x + 9 + y^2 = 9 \quad \Longrightarrow \quad \boxed{x^2 - 6x + y^2 = 0}$$

b) The distance from $(5, 2)$ to $(3, 0)$ is

$$\sqrt{(5-3)^2 + (2-0)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

This is smaller than the radius (because $\sqrt{8} < \sqrt{9}$) which means that $(5, 2)$ lies inside the circle.

Exercise 6.1.

a) $-\ln\frac{1}{7} = \ln\left(\left(\frac{1}{7}\right)^{-1}\right) = \boxed{\ln 7}$

b) $\ln 6 - \ln 3 = \ln\frac{6}{3} = \boxed{\ln 2}$

c) $\frac{\ln 9}{\ln 3} = \frac{\ln 3^2}{\ln 3} = \frac{2\ln 3}{\ln 3} = \boxed{2}$

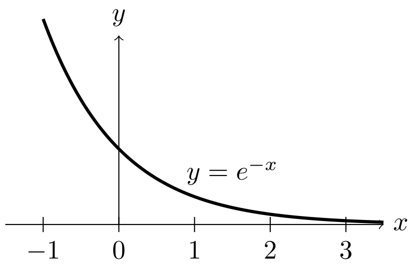
d) $\ln 2 + \ln 0.5 = \ln(2\cdot 0.5) = \ln 1 = \boxed{0}$

Exercise 6.2. ${}^{10}\log 0.001 = -3$, because $10^{-3} = 0.001$.

Exercise 6.3. The desired intersection point (x, y) satisfies $e^{-x} = y$ and $y = 0.2$:

$$e^{-x} = 0.2 \quad\Longrightarrow\quad -x = \ln 0.2 \quad\Longrightarrow\quad x = -\ln 0.2 = \ln 0.2^{-1} = \ln 5$$

Conclusion: the intersection point is $(\ln 5\,,\, 0.2)$.



Exercise 6.4. Let's free the exponents by taking the ln of both sides of the equation:

$$5^{x+1} = 7^{x-1} \quad\Longrightarrow\quad \ln 5^{x+1} = \ln 7^{x-1} \quad\Longrightarrow\quad (x+1)\ln 5 = (x-1)\ln 7$$
$$\Longrightarrow\quad x\ln 7 - x\ln 5 = \ln 7 + \ln 5 \quad\Longrightarrow\quad x = \frac{\ln 7 + \ln 5}{\ln 7 - \ln 5} = \frac{\ln 35}{\ln\frac{7}{5}}$$

Exercise 6.5. Silvia and Luke are equally rich when $\sqrt{e^t} = 2^{1-t}$. I am going to take the ln on both sides of this equation to solve for t:

$$\ln\left(\sqrt{e^t}\right) = \ln 2^{1-t} \quad\Longrightarrow\quad \frac{1}{2}\ln e^t = (1-t)\ln 2 \quad\Longrightarrow\quad \frac{1}{2}t = (1-t)\ln 2 \quad\Longrightarrow\quad \frac{1}{2}t = \ln 2 - t\ln 2$$
$$\Longrightarrow\quad \frac{1}{2}t + t\ln 2 = \ln 2 \quad\Longrightarrow\quad \left(\frac{1}{2} + \ln 2\right)t = \ln 2 \quad\Longrightarrow\quad t = \frac{\ln 2}{\frac{1}{2} + \ln 2} = \frac{\ln 4}{1 + \ln 4}$$

Exercise 6.6. $\left(\sqrt{e}\right)^{\ln 9} = \left(e^{\frac{1}{2}}\right)^{\ln 9} = e^{\frac{1}{2}\ln 9} = e^{\ln 3} = 3$

Exercise 6.7. $e^{3\ln t} = e^{\ln t^3} = t^3$

Exercise 6.8. $\ln(2e^2 + e^2) = \ln(3e^2) = \ln(3) + \ln(e^2) = \ln(3) + 2$

Note that this is NOT the same as $\ln(2e^2 + e^2) \neq \ln(2e^2) + \ln(e^2) = 2\ln(2) + 2$.

Exercise 6.9. $49^t = e^{\ln\left(49^t\right)} = e^{t\ln 49} = e^{2t\ln 7} = \boxed{e^2}$

If you prefer to tackle this problem with the first ln-trick, you can proceed as follows: let's first call the desired number x, so we have $x = 49^t$. Now, take the ln of both sides and substitute $t = \frac{1}{\ln 7}$:

$$\ln x = \ln 49^t = t\ln 49 = \frac{1}{\ln 7}\cdot\ln 49 = \frac{\ln 49}{\ln 7} = \frac{\ln(7^2)}{\ln 7} = \frac{2\ln 7}{\ln 7} = 2$$

Finally, calculate x by taking the exp of both sides:

$$\ln x = 2 \quad\Longrightarrow\quad e^{\ln x} = e^2 \quad\Longrightarrow\quad \boxed{x = e^2}$$

Exercise 6.10. You can use the relationship between ${}^a\!\log$ and ln for this:

$$\boxed{{}^a\!\log x = \frac{\ln x}{\ln a}}$$

Conclusion: ${}^2\!\log x = \dfrac{\ln x}{\ln 2}$

Exercise 6.11. This is a matter of properly using the calculation rules:

$$\ln 2x = 1 + \ln x^2 \implies \ln 2 + \ln x = 1 + 2\ln x \implies \ln x = \ln 2\, - 1$$
$$\implies x = e^{\ln 2\,-1} = e^{\ln 2}\cdot e^{-1} = \frac{2}{e}$$

Exercise 6.12. At first glance, this looks like an ordinary equation with exponentials, so you might be tempted to take the ln of both sides to free the x's in the exponents. This will however not bring you any further, because you cannot simplify $\ln(20+e^x)$. Moving terms to the other side of this equation also doesn't help, so what should we do?

The important insight here is that $e^{2x} = (e^x)^2$, so this equation is a quadratic equation in e^x. You can see this more clearly by substituting $p = e^x$ into the equation:

$$e^{2x} = 20 + e^x \xRightarrow{\text{substitute } p = e^x} p^2 = 20 + p \implies (p-5)(p+4) = 0 \implies p = 5 \text{ or } p = -4$$

This means that either $e^x = 5$, which yields the solution $\boxed{x = \ln 5}$, or $e^x = -4$, which yields no solutions since a positive number to the power of another number can never be negative.

Exercise 6.13. This is the case when $3^t = 1000$. You can solve this equation by taking the ln of both sides:

$$\ln(3^t) = \ln 1000 \implies t\ln 3 = \ln 1000 = \ln(10^3) = 3\ln 10 \implies t = \frac{3\ln 10}{\ln 3}$$

Exercise 6.14.

- The first one is easy: start with $e^0 = 1$ and take the ln of both sides:

$$e^0 = 1 \xRightarrow{\text{take ln}} \ln(e^0) = \ln(1) \xRightarrow{\text{apply } \ln(e^x)=x} 0 = \ln(1)$$

- The second one is much trickier. We will need to use a clever intermediate step: we're going to use (6.3) and write x as $e^{\ln(x)}$. Similarly, we can write y as $e^{\ln(y)}$. Now, if we take the product of x and y and apply the calculation rule $e^a \cdot e^b = e^{a+b}$, we get

$$xy = e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x)+\ln(y)}$$

  and if we take the ln of both sides of this equation, we obtain

$$\ln(xy) = \ln\left(e^{\ln(x)+\ln(y)}\right).$$

  This looks terrible, but remember that $\ln(e^{\text{something}}) = \text{something}$, so we can simplify the right-hand side to obtain

$$\ln(xy) = \ln(x) + \ln(y)$$

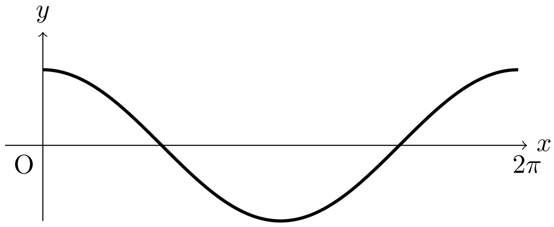
- The third rule works similarly to the second rule:

$$\frac{x}{y} = \frac{e^{\ln(x)}}{e^{\ln(y)}} = e^{\ln(x)-\ln(y)} \xRightarrow{\text{take ln}} \ln\left(\frac{x}{y}\right) = \ln\left(e^{\ln(x)-\ln(y)}\right) = \ln(x) - \ln(y)$$

- The fourth rule needs a similar trick: write x as $e^{\ln(x)}$ and use $(e^a)^b = e^{ab}$, yielding

$$x^y = (e^{\ln(x)})^y = e^{\ln(x)\cdot y} = e^{y\ln(x)} \xRightarrow{\text{take ln}} \ln(x^y) = \ln\left(e^{y\ln(x)}\right) = y\ln(x)$$

Exercise 7.1. This is the graph of $y = \cos(x)$ on the domain $0 \le x \le 2\pi$:



Exercise 7.2.

30 degrees	$=$	$\frac{\pi}{6}$ radians
$\frac{5\pi}{4}$ radians	$=$	225 degrees
63 degrees	$=$	$\frac{7}{20}\pi$ radians

Exercise 7.3. The whole unit circle has an arc length of 2π and a surface area of π. An arc length of 1 is $\frac{1}{2\pi}$ times the arc length of the whole unit circle, so the corresponding surface area is also $\frac{1}{2\pi}$ times the surface area of the whole unit circle:

$$\frac{1}{2\pi} \cdot \pi = \frac{1}{2}$$

Exercise 7.4.

a) 150 degrees is equal to $\dfrac{150\pi}{180}$ radians, which is equal to $\dfrac{5\pi}{6}$, so the sine of 150 degrees is equal to $\sin(\frac{5}{6}\pi)$.
Let's calculate this using the calculation rule $\sin(\pi - x) = \sin x$:

$$\sin(\tfrac{5}{6}\pi) = \sin(\pi - \tfrac{1}{6}\pi) = \sin(\tfrac{1}{6}\pi) = \boxed{\frac{1}{2}}$$

Alternatively, you can draw the unit circle, look for the point on the unit circle with angle $\frac{5}{6}\pi$ with the positive x-axis, and deduce its y-coordinate to find $\sin(\frac{5}{6}\pi)$.

b) The calculation rule $\cos(x + 2n\pi) = \cos x$ tells you that you can add any multiple of 2π to the argument of cos (because cos is periodic with period 2π). Together with $\cos(\pi - x) = -\cos x$ this turns out to be very useful here:

$$\cos\left(-\frac{37}{4}\pi\right) = \cos\left(-\frac{37}{4}\pi + 10\pi\right) = \cos\frac{3}{4}\pi = \cos\left(\pi - \frac{1}{4}\pi\right) = -\cos\frac{1}{4}\pi = \boxed{-\frac{1}{2}\sqrt{2}}$$

Of course, if you used the unit circle to find $\cos\frac{3}{4}\pi$ directly, that is also perfectly fine.

Exercise 7.5. Using the unit circle, you should be able to find:

a) $\sin\dfrac{2\pi}{3} = \dfrac{1}{2}\sqrt{3}$

b) $\tan\left(-\dfrac{3\pi}{4}\right) = \dfrac{\sin(-\frac{3}{4}\pi)}{\cos(-\frac{3}{4}\pi)} = \dfrac{-\frac{1}{2}\sqrt{2}}{-\frac{1}{2}\sqrt{2}} = 1$

c) $\tan\dfrac{\pi}{2} = \dfrac{\sin(\frac{1}{2}\pi)}{\cos(\frac{1}{2}\pi)} = \dfrac{1}{0} = \text{undefined}$

d) $\sin\dfrac{7\pi}{6} = -\dfrac{1}{2}$

Exercise 7.6. You can use the following calculation rules for these exercises:

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

If $\cos x = 0.9$ then

$$\cos 2x = 2\cos^2 x - 1 = 2(0.9)^2 - 1 = \boxed{0.62}$$

and if $\sin x = 0.3$ then

$$\cos 2x = 1 - 2\sin^2 x = 1 - 2(0.3)^2 = \boxed{0.82}$$

Exercise 7.7.

a) The unit circle tells us that if $\sin\alpha = \sin\beta$, we have two options:

- $\alpha = \beta$ plus any multiple of 2π
- $\alpha = \pi - \beta$ plus any multiple of 2π

If we apply this rule to $\sin 2x = \sin(x + \frac{1}{2}\pi)$, we have to solve the following two equations:

- $2x = x + \frac{1}{2}\pi$ plus any multiple of 2π, so

$$2x = x + \frac{\pi}{2} + 2k\pi \text{ with } k \text{ an integer} \quad\Longleftrightarrow\quad x = \frac{\pi}{2} + 2k\pi$$

This gives us a whole range of values for x (for each integer k a different one), but the only solution on the interval $[0, 2\pi]$ is $x = \frac{1}{2}\pi$.

- $2x = \pi - (x + \frac{1}{2}\pi)$ plus any multiple of 2π, so

$$2x = \pi - x - \frac{\pi}{2} + 2k\pi \text{ with } k \text{ an integer} \quad\Longleftrightarrow\quad 3x = \frac{\pi}{2} + 2k\pi \quad\Longleftrightarrow\quad x = \frac{\pi}{6} + \frac{2k\pi}{3}$$

This gives us $x = \frac{1}{6}\pi$, $x = \frac{5}{6}\pi$ and $x = \frac{3}{2}\pi$ on the interval $[0, 2\pi]$.

Conclusion: $\boxed{x = \frac{\pi}{6}}$, $\boxed{x = \frac{\pi}{2}}$, $\boxed{x = \frac{5\pi}{6}}$ or $\boxed{x = \frac{3\pi}{2}}$.

b) There exist no rules to solve equations of the form $\sin\alpha = \cos\beta$ directly, but we can use $\cos\beta = \sin(\frac{1}{2}\pi - \beta)$ to transform the cosine into a sine:

$$\sin 3x = \cos 2x = \sin\left(\frac{\pi}{2} - 2x\right) \quad\Longleftrightarrow\quad 3x = \frac{\pi}{2} - 2x + 2k\pi \text{ with } k \in \mathbb{Z} \text{ or } 3x = \pi - \left(\frac{\pi}{2} - 2x\right) + 2k\pi \text{ with } k \in \mathbb{Z}$$

The first option yields $5x = \frac{1}{2}\pi + 2k\pi$ so $x = \frac{1}{10}\pi + \frac{2}{5}k\pi$ and the second option yields $x = \frac{1}{2}\pi + 2k\pi$.

The solutions on $[-\pi, \pi]$ are therefore $\boxed{x = -\frac{7\pi}{10}}$, $\boxed{x = -\frac{3\pi}{10}}$, $\boxed{x = \frac{\pi}{10}}$, $\boxed{x = \frac{\pi}{2}}$ and $\boxed{x = \frac{9\pi}{10}}$.

Exercise 7.8.

a) $(\sin t + \cos t)^2 - \sin 2t = \sin^2 t + 2\sin t\cos t + \cos^2 t - \sin 2t = 1$

(Used calculation rules: $(p+q)^2 = p^2 + 2pq + q^2$ and $\sin^2 t + \cos^2 t = 1$ and $\sin 2t = 2\sin t\cos t$)

b) $\cos^4 t - \sin^4 t = \left(\cos^2 t + \sin^2 t\right)\left(\cos^2 t - \sin^2 t\right) = \cos 2t$

(Used calculation rules: $p^2 - q^2 = (p+q)(p-q)$ and $\sin^2 t + \cos^2 t = 1$ and $\cos 2t = \cos^2 t - \sin^2 t$)

c) $\cos 7t\cos 2t - \sin 7t\sin 2t = \cos(7t + 2t) = \cos 9t$

(Used calculation rule: $\cos(x+y) = \cos x\cos y - \sin x\sin y$)

Exercise 7.9.

a) Since sin has period 2π, you can freely add or subtract any multiple of 2π from the argument of the sine:

$$\sin(2\pi - x) = \sin(-x) = \boxed{-\sin x}$$

b) This is where the identity $\cos(x - y) = \cos x\cos y + \sin x\sin y$ turns out to be useful:

$$\cos\left(\frac{3\pi}{2} - x\right) = \cos\frac{3\pi}{2}\cos x + \sin\frac{3\pi}{2}\sin x = 0\cdot\cos x - 1\cdot\sin x = \boxed{-\sin x}$$

c) Using $\sin(x+y) = \sin x\cos y + \cos x\sin y$ we get:

$$\sin\left(\frac{3\pi}{4} + x\right) = \sin\frac{3\pi}{4}\cos x + \cos\frac{3\pi}{4}\sin x = \frac{1}{2}\sqrt{2}\cdot\cos x - \frac{1}{2}\sqrt{2}\cdot\sin x = \boxed{\frac{\sqrt{2}}{2}(\cos x - \sin x)}$$

d)

$$\tan x + \frac{1}{\tan x} = \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x}{\sin x\cos x} + \frac{\cos^2 x}{\cos x\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x\cos x} = \boxed{\frac{1}{\sin x\cos x}}$$

Exercise 7.10. This is a quadratic equation in disguise, which you can see if you apply the calculation rule $\cos 2x = 2\cos^2 x - 1$:

$$3\cos 2x + 5\cos x = 3 \xRightarrow{\cos 2x = 2\cos^2 x - 1} 3(2\cos^2 x - 1) + 5\cos x = 3 \implies 6\cos^2 x + 5\cos x - 6 = 0$$

Let's substitute $p = \cos x$ and divide the whole lot by 6, so that we can hopefully factor the quadratic equation:

$$6\cos^2 x + 5\cos x - 6 = 0 \xRightarrow{p = \cos x} 6p^2 + 5p - 6 = 0 \xRightarrow{\text{divide by 6}} p^2 + \frac{5}{6}p - 1 = 0$$

That wasn't really helpful... You will learn how to solve quadratic equations in the next chapter, so for now I will just tell you that you can factor this equation as follows:

$$p^2 + \frac{5}{6}p - 1 = \left(p - \frac{2}{3}\right)\left(p + \frac{3}{2}\right) = 0 \implies p = \frac{2}{3} \text{ or } p = -\frac{3}{2} \xRightarrow{p = \cos x} \cos x = \frac{2}{3} \text{ or } \cos x = -\frac{3}{2}$$

The second option doesn't make sense, because cos is always between -1 and 1, so the first option is our only solution:

$$\boxed{\cos x = \frac{2}{3}}$$

Exercise 7.11. One possibility is using the formula $\cos 2x = 1 - 2\sin^2 x$, with $x = \frac{\pi}{12}$:

$$\cos\frac{\pi}{6} = 1 - 2\left(\sin\frac{\pi}{12}\right)^2 \implies 2\left(\sin\frac{\pi}{12}\right)^2 = 1 - \frac{\sqrt{3}}{2} \implies \left(\sin\frac{\pi}{12}\right)^2 = \frac{2-\sqrt{3}}{4}$$
$$\implies \sin\frac{\pi}{12} = \pm\frac{1}{2}\sqrt{2-\sqrt{3}} \implies \sin\frac{\pi}{12} = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

A much easier solution:

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

You might wonder: are these two solutions actually the same? Of course they are, but you will need to work your roots tricks hard in order to show this. Let's first rewrite $\sqrt{6} - \sqrt{2}$ a bit:

$$\sqrt{6} - \sqrt{2} = \sqrt{3\cdot 2} - \sqrt{2} = \sqrt{3}\sqrt{2} - \sqrt{2} = \left(\sqrt{3} - 1\right)\sqrt{2}$$

It's time for even more magic: I am going to square $\sqrt{3} - 1$ and then take the square root of that square. That essentially changes nothing, but I will be able to rewrite $\sqrt{3} - 1$ quite drastically:

$$\sqrt{3} - 1 = \sqrt{(\sqrt{3}-1)^2} = \sqrt{(\sqrt{3})^2 - 2\sqrt{3} + 1} = \sqrt{3 - 2\sqrt{3} + 1} = \sqrt{4 - 2\sqrt{3}} = \sqrt{2(2-\sqrt{3})} = \sqrt{2}\cdot\sqrt{2-\sqrt{3}}$$

Now we're getting somewhere, because if I plug this into the equation we started with, I get

$$\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{(\sqrt{3}-1)\sqrt{2}}{4} = \frac{\sqrt{2}\cdot\sqrt{2-\sqrt{3}}\cdot\sqrt{2}}{4} = \frac{2\sqrt{2-\sqrt{3}}}{4} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Voilà!

Exercise 7.12.

a) Let's split this problem into two parts:

- $\sin 3t = \sin(2t + t) = \sin t\cos 2t + \cos t\sin 2t = \sin t(2\cos^2 t - 1) + \cos t(2\sin t\cos t)$
  $= 2\sin t\cos^2 t - \sin t + 2\sin t\cos^2 t = 4\sin t\cos^2 t - \sin t$
- $4\sin^3 t = 4\sin t\sin^2 t = 4\sin t(1 - \cos^2 t) = 4\sin t - 4\sin t\cos^2 t$

Conclusion:

$$\sin 3t + 4\sin^3 t = 4\sin t\cos^2 t - \sin t + 4\sin t - 4\sin t\cos^2 t = 3\sin t$$

b) $\dfrac{\sin 2t - \tan t}{\tan t} = \dfrac{2\sin t\cos t - \frac{\sin t}{\cos t}}{\frac{\sin t}{\cos t}} = \dfrac{2\sin t\cos^2 t - \sin t}{\sin t} = 2\cos^2 t - 1 = \cos 2t$

Exercise 7.13. $1+\tan^2 x = \dfrac{\cos^2 x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

Exercise 8.1.

a) $\dfrac{3}{2}x + \dfrac{5}{7} = 1 - x \quad \overset{\cdot 14}{\Longrightarrow} \quad 21x + 10 = 14 - 14x \quad \overset{\text{sort}}{\Longrightarrow} \quad 35x = 4 \quad \Longrightarrow \quad x = \dfrac{4}{35}$

b) $-2(x-5) = 3(2-3x) + 5(1-x) \quad \Longrightarrow \quad -2x + 10 = 6 - 9x + 5 - 5x \quad \Longrightarrow \quad 12x = 1 \quad \Longrightarrow \quad x = \dfrac{1}{12}$

c) $\dfrac{1-4x}{x-3} = 5 \quad \Longrightarrow \quad 1 - 4x = 5(x-3) \quad \Longrightarrow \quad 1 - 4x = 5x - 15 \quad \Longrightarrow \quad 16 = 9x \quad \Longrightarrow \quad x = \dfrac{16}{9}$

Exercise 8.2. Suppose I add x litres of wine. The percentage of alcohol then becomes

$$\frac{\text{amount of alcohol}}{\text{amount of mixed drink}} \cdot 100\% = \frac{0.12\,x + 0.05}{x+1} \cdot 100\% = \frac{12x+5}{x+1} \text{ percent}$$

so the equation to solve becomes

$$\frac{12x+5}{x+1} = 8 \quad \Longrightarrow \quad 12x + 5 = 8(x+1) \quad \Longrightarrow \quad 12x + 5 = 8x + 8 \quad \Longrightarrow \quad 4x = 3 \quad \Longrightarrow \quad x = \frac{3}{4}$$

So I have to add 0.75 litres of wine to the beer.

Exercise 8.3.

$$1 < \frac{7}{2} - \frac{1}{6}x < 2 \quad \overset{\cdot 6}{\Longleftrightarrow} \quad 6 < 21 - x < 12 \quad \overset{-21}{\Longleftrightarrow} \quad -15 < -x < -9 \quad \overset{\cdot(-1)}{\Longleftrightarrow} \quad 15 > x > 9$$

So the solutions are all real numbers between 9 and 15.

Exercise 8.4.

a) $x^2 = 1 + 2x \quad \Longrightarrow \quad x^2 - 2x - 1 = 0 \quad \xrightarrow{\text{quadratic formula}} \quad x = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

b) $x^2 = 6x + 7 \quad \Longrightarrow \quad x^2 - 6x - 7 = 0 \quad \xrightarrow{\text{factorise}} \quad (x-7)(x+1) = 0 \quad \Longrightarrow \quad x = 7 \text{ of } -1$

c) $3x^2 + 5x = (x+1)^2 \quad \Longrightarrow \quad 3x^2 + 5x = x^2 + 2x + 1 \quad \Longrightarrow \quad 2x^2 + 3x - 1 = 0 \quad \Longrightarrow \quad x^2 + \dfrac{3}{2}x - \dfrac{1}{2} = 0$

$$\xrightarrow{\text{completing the square}} \quad \left(x + \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{1}{2} = 0 \quad \Longrightarrow \quad x + \frac{3}{4} = \pm\frac{\sqrt{17}}{4} \quad \Longrightarrow \quad x = -\frac{3}{4} \pm \frac{\sqrt{17}}{4}$$

Exercise 8.5. You should rewrite $x^2 + 3x$ as $(x + \frac{3}{2})^2 = x^2 + 3x + \frac{9}{4}$ minus the additional $\frac{9}{4}$:

$$x^2 + 3x + 1 = 0 \quad \Longrightarrow \quad \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 1 = 0 \quad \Longrightarrow \quad \left(x + \frac{3}{2}\right)^2 = \frac{5}{4} \quad \Longrightarrow \quad x + \frac{3}{2} = \pm\sqrt{\frac{5}{4}}$$

Conclusion: the solutions are $-\frac{3}{2} + \frac{1}{2}\sqrt{5}$ and $-\frac{3}{2} - \frac{1}{2}\sqrt{5}$.

Exercise 8.6. Because $(x - \frac{5}{2})^2 = x^2 - 5x + \frac{25}{4}$ we can write $5x - x^2$ as

$$5x - x^2 = \frac{25}{4} - \left(x - \frac{5}{2}\right)^2$$

Conclusion: the maximum value of $5x - x^2$ is $\frac{25}{4}$ because the term $-(\cdots)^2$ is either negative or zero (zero at $x = \frac{5}{2}$).

Exercise 8.7. Let's square left and right first:

$$2\sin t = \sqrt{5 - 4\cos t} \quad \Longrightarrow \quad 4\sin^2 t = 5 - 4\cos t$$

This is a quadratic equation in $\cos t$, which you can see if you rewrite $\sin^2 t$ as $1 - \cos^2 t$, because the equation then becomes $4\left(1 - \cos^2 t\right) = 5 - 4\cos t$ which you can sort to

$$4\cos^2 t - 4\cos t + 1 = 0 \quad \Longrightarrow \quad (2\cos t - 1)^2 = 0 \quad \Longrightarrow \quad \cos t = \frac{1}{2} \quad \Longrightarrow \quad t = \frac{\pi}{3}$$

Exercise 8.8.

$x = 2\sqrt{x}+3 \iff x-2\sqrt{x}-3 = 0 \iff (\sqrt{x})^2-2\sqrt{x}-3 = 0 \iff (\sqrt{x}-3)(\sqrt{x}+1) = 0 \iff \sqrt{x} = 3$ or $\sqrt{x} = -1$

$\sqrt{x}$ cannot be negative, so we can conclude that $\sqrt{x} = 3$ and therefore $x = 9$.

Exercise 8.9. Their positions are equal when

$$e^{2t} + e^t - 6 = 0 \iff (e^t - 2)(e^t + 3) = 0 \iff e^t = 2 \iff \boxed{t = \ln 2}$$

Exercise 8.10. By completing the square I can write the equation of the circle in its standard form:

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{2} \implies \begin{cases} \text{the centre is } \left(\frac{3}{2}, \frac{1}{2}\right) \\ \text{the radius is } \sqrt{\frac{5}{2}} \end{cases}$$

So the surface area is $\pi \cdot \text{radius}^2 = \frac{5}{2}\pi$.

Exercise 8.11. I eliminate y as follows:

$$\begin{array}{rcrl} 2x + y = 7 & \overset{\cdot 3}{\implies} & 6x + 3y = 21 & \\ & & 5x - 3y = \ \ 1 & \\ & + & \hline 11x \qquad = 22 & \implies x = 2 \implies y = 3 \end{array}$$

Exercise 8.12. I substitute $y = x - 3$ into $x = y^2 + 1$:

$$x = (x-3)^2 + 1 \iff x = x^2 - 6x + 9 + 1 \iff x^2 - 7x + 10 = 0 \iff (x-5)(x-2) = 0$$

So we have $x = 5$ or $x = 2$ and we can use $y = x - 3$ to find the corresponding y-values:

- $x = 5$ yields $y = 2$
- $x = 2$ yields $y = -1$

Exercise 8.13.

$$\frac{e^t}{1 + e^{2t}} = 0.3 \iff \frac{e^t}{1 + e^{2t}} = \frac{3}{10} \iff 10e^t = 3 + 3e^{2t} \iff \left(e^t\right)^2 - \frac{10}{3}e^t + 1 = 0$$

Let's complete the square:

$$\left(e^t - \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0 \iff \left(e^t - \frac{5}{3}\right)^2 = \frac{16}{9} \iff e^t - \frac{5}{3} = \pm\frac{4}{3} \iff e^t = \frac{5}{3} \pm \frac{4}{3}$$

Conclusion: $e^t = 3$ or $e^t = \frac{1}{3}$ and this means that $t = \ln 3$ or $t = \ln(\frac{1}{3}) = -\ln 3$.

Exercise 9.1. $\frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

Exercise 9.2.

a) $\frac{d}{dx}x^2 = 2x$

b) The slope in the point $(2, 4)$ is $\left[\frac{d}{dx}x^2\right]_{x=2} = \Big[2x\Big]_{x=2} = 4$.

Exercise 9.3. You should start by calculating the relevant slope:

$$xy = 4 \implies y = \frac{4}{x} \implies \frac{dy}{dx} = -\frac{4}{x^2} \implies \left[\frac{dy}{dx}\right]_{x=1} = -4$$

This implies that the tangent line can be expressed as $y = -4x + b$. Because $(1, 4)$ has to satisfy this equation, we have $b = 8$, so the equation of the tangent line is

$$\boxed{4x + y = 8}$$

Exercise 9.4.

a) $\frac{d}{dx} 6x\sqrt[3]{x} = 6 \cdot \frac{d}{dx} x^{\frac{4}{3}} = 6 \cdot \frac{4}{3} x^{\frac{1}{3}} = 8\sqrt[3]{x}$

b) $\frac{d}{dx}\left(\frac{x^2}{7} - \frac{7}{x^2}\right) = \frac{1}{7} \cdot \frac{d}{dx} x^2 - 7 \cdot \frac{d}{dx} x^{-2} = \frac{1}{7} \cdot 2x - 7 \cdot \left(-2x^{-3}\right) = \frac{2}{7}x + \frac{14}{x^3}$

c) $\frac{d}{dx} \frac{x^5 - 3}{x} = \frac{d}{dx}\left(x^4 - \frac{3}{x}\right) = 4x^3 + \frac{3}{x^2}$

Exercise 9.5. You can find the derivative of this function using the quotient rule:

$$\frac{d}{dx} f(x) = \frac{x \cdot (4x - 5) - (2x^2 - 5x + 4) \cdot 1}{x^2} = \frac{2x^2 - 4}{x^2}$$

This derivative is zero where the function has a minimum:

$$\frac{2x^2 - 4}{x^2} = 0 \implies 2x^2 - 4 = 0 \implies x^2 = 2 \implies x = \sqrt{2}$$

So the minimum value is $f(\sqrt{2}) = 4\sqrt{2} - 5$.

Exercise 9.6.

a) Quotient rule $\implies \frac{d}{dx} \frac{x^2 - 1}{x^2 + 1} = \frac{2x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$

b) Chain rule $\implies \frac{d}{dx} (1 + 3x)^5 = 5(1 + 3x)^4 \cdot 3 = 15(1 + 3x)^4$

c) Chain rule $\implies \frac{d}{dx} \ln(1 + \sqrt{x}) = \frac{1}{1 + \sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2(\sqrt{x} + x)} = \frac{x - \sqrt{x}}{2x^2 - 2x}$

Exercise 9.7. I apply the chain rule with $s = \sqrt{u}$ and $u = t^3 + t^2$:

$$v = \frac{ds}{dt} = \frac{ds}{du} \cdot \frac{du}{dt} = \frac{1}{2\sqrt{u}} \cdot (3t^2 + 2t) = \frac{3t^2 + 2t}{2\sqrt{t^3 + t^2}} = \frac{3t + 2}{2\sqrt{t + 1}}$$

Exercise 9.8.

a) If we define $R(t)$ to be the rate at which my cat's weight increases per unit time, then $R(t)$ is given by the derivative of the weight function with respect to time:

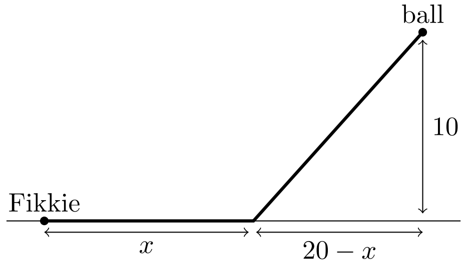
$$R(t) = \frac{dW}{dt} = \frac{1}{1 + t^2} \cdot 2t = \frac{2t}{1 + t^2}$$

b) My cat is gaining weight fastest when its weight gaining rate $R(t)$ is maximum, which is the case when $R'(t)$ is zero, so we have to differentiate $R(t)$, which we can do using the quotient rule:

$$\frac{dR}{dt} = \frac{(1 + t^2) \cdot 2 - 2t \cdot 2t}{(1 + t^2)^2} = \frac{2 + 2t^2 - 4t^2}{(1 + t^2)^2} = \frac{2 - 2t^2}{(1 + t^2)^2}$$

$R'(t)$ is equal to zero when $2 - 2t^2 = 0$, so $t = -1$ (which doesn't make sense) or $t = 1$. Conclusion: my cat will be gaining weight fastest after one year.

Exercise 9.9. Let's define x to be Fikkie's running distance:



We can now express the time t Fikkie takes to reach its ball in terms of x as follows: the running takes $t_{\text{run}} = x/v_{\text{run}}$ and the swimming takes $t_{\text{swim}} = d_{\text{swim}}/v_{\text{swim}}$ with $d_{\text{swim}} = \sqrt{(20-x)^2+10^2}$. Let's put all the pieces together:

$$t = t_{\text{run}} + t_{\text{swim}} = \frac{x}{v_{\text{run}}} + \frac{d_{\text{swim}}}{v_{\text{swim}}} = \frac{x}{3} + \frac{\sqrt{(20-x)^2+100}}{2}$$

This time is minimum when dt/dx is zero:

$$\frac{dt}{dx} = \frac{1}{3} + \frac{-2(20-x)}{4\sqrt{(20-x)^2+100}} = \frac{1}{3} - \frac{20-x}{2\sqrt{(20-x)^2+100}} = 0 \implies \frac{20-x}{2\sqrt{(20-x)^2+100}} = \frac{1}{3}$$

$$\xRightarrow{\text{cross-multiplication}} 3(20-x) = 2\sqrt{(20-x)^2+100} \xRightarrow{\text{square both sides}} 9(20-x)^2 = 4(20-x)^2+400$$

$$\implies 5(20-x)^2 = 400 \implies (20-x)^2 = 80 \implies 20-x = \pm\sqrt{80} = \pm 4\sqrt{5} \implies x = 20 \pm 4\sqrt{5}$$

The plus sign makes no sense, because $0 \le x \le 20$, so Fikkie should run $\boxed{20 - 4\sqrt{5}}$ metres before jumping into the water.

Exercise 9.10. I will give you the derivatives and mention the rules to be used:

a) $\sin x + x\cos x$ (product rule)

b) $\dfrac{2}{(1+x)^2}$ (quotient rule)

c) $e^x(x+1)$ (product rule)

d) $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$ (chain rule)

e) $\dfrac{-4x^2+4x+4}{(x^2+1)^2}$ (quotient rule)

f) $3(\cos x - \sin x)(\sin x + \cos x)^2$ (chain rule)

g) $(1+e^x)\cos(x+e^x)$ (chain rule)

h) $\dfrac{1+\cos x}{x+\sin x}$ (chain rule)

i) $-\dfrac{2\sin x\cos x}{1+\cos^2 x}$ (twice the chain rule)

j) $3x^2 e^{(x^3)}$ (chain rule)

Exercise 10.1. You can calculate these antiderivatives using the standard antiderivative $\displaystyle\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}$. The correct answers are:

a) $\frac{4}{3}x^3 + C$ b) $\frac{5}{88}x^{11} + C$ c) $\frac{7}{15}x^5 + C$ d) $\frac{2}{3}\sqrt{2}\cdot x^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{2}\cdot x\sqrt{x} + C$ e) $-\frac{5}{4x} + C$

Exercise 10.2. Rewrite $\sqrt{\cdots}$ as $(\cdots)^{\frac{1}{2}}$ and use the standard derivative $\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}$. Don't forget the reverse chain rule (RCR)!

a) $\int 3\sqrt{x+5}\,dx = 3\int (x+5)^{\frac{1}{2}}\,dx = 3\cdot\frac{2}{3}(x+5)^{\frac{3}{2}} + C = 2(x+5)\sqrt{x+5} + C$

b) $\int \frac{1}{\sqrt{3t+2}}\,dt = \int (3t+2)^{-\frac{1}{2}}\,dt = 2(3t+2)^{\frac{1}{2}} \underbrace{\cdot\frac{1}{3}}_{\text{RCR}} + C = \frac{2}{3}\sqrt{3t+2} + C$

c) $\int \sqrt{13-\alpha}\,d\alpha = \int (13-\alpha)^{\frac{1}{2}}\,d\alpha = \frac{2}{3}(13-\alpha)^{\frac{3}{2}} \cdot \underbrace{\frac{1}{-1}}_{\text{RCR}} + C = -\frac{2}{3}(13-\alpha)\sqrt{13-\alpha} + C = \frac{2}{3}(\alpha-13)\sqrt{13-\alpha} + C$

Exercise 10.3.

a) $\int e^{3x}\,dx = \underbrace{\frac{1}{3}}_{\text{RCR}} \cdot e^{3x} + C$

b) $\int \frac{2}{e^{2t}}\,dt = 2\int e^{-2t}\,dt = 2\cdot \underbrace{\frac{1}{-2}}_{\text{RCR}} \cdot e^{-2t} + C = -e^{-2t} + C = -\frac{1}{e^{2t}} + C$

c) $\int \frac{e^x - e^{-x}}{2}\,dx = \int \frac{e^x}{2}\,dx - \int \frac{e^{-x}}{2}\,dx = \frac{e^x}{2} - \underbrace{\frac{1}{-1}}_{\text{RCR}} \cdot \frac{e^{-x}}{2} + C = \frac{e^x + e^{-x}}{2} + C$

d) $\int \frac{5}{6y}\,dy = \frac{5}{6}\int \frac{1}{y}\,dy = \frac{5}{6}\ln|y| + C$

Exercise 10.4.

a) $\int \cos(\pi x)\,dx = \underbrace{\frac{1}{\pi}}_{\text{RCR}} \cdot \sin(\pi x) + C$

b) $\int \sin(ax+b)\,dx = \underbrace{\frac{1}{a}}_{\text{RCR}} \cdot -\cos(ax+b) + C = -\frac{1}{a}\cos(ax+b) + C$

c) $\int \sqrt{2}\cos(1-x)\,dx = \sqrt{2}\cdot \underbrace{\frac{1}{-1}}_{\text{RCR}} \cdot \sin(1-x) + C = -\sqrt{2}\sin(1-x) + C$

Exercise 10.5. These antiderivatives cannot be calculated directly, but we can cleverly use some of the trigonometric identities from chapter 7:

a) $\int 2\sin x\cos x\,dx = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$

b) $\int \left(2x\sin^2 x + 2x\cos^2 x\right)dx = \int 2x\left(\sin^2 x + \cos^2 x\right)dx = \int 2x\cdot 1\,dx = x^2 + C$

c) $\int 3\cos x\tan x\,dx = \int 3\cos x\,\frac{\sin x}{\cos x}\,dx = 3\int \sin x\,dx = -3\cos x + C$

Exercise 10.6. Let's see what happens if we differentiate $F(x) = x\ln x$ using the product rule:

$$F'(x) = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$$

This is almost equal to $f(x) = \ln x$, but we have to get rid of the $+1$, which we can do by adding $-x$ to our tentative antiderivative:

$$F(x) = x\ln x - x \quad \Longrightarrow \quad F'(x) = \ln x + 1 - 1 = \ln x = f(x)$$

Conclusion:

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

Exercise 10.7.

$$f_3(x) = \frac{4\,(x+9)\,(x+3)}{x^2+12x+27} = \frac{4(x^2+12x+27)}{x^2+12x+27} = 4 \quad \Longrightarrow \quad \int f_3(x)\,dx = \int 4\,dx = 4x + C$$

Exercise 10.8. I know that I can find my velocity $v(t)$ by differentiating the distance $s(t)$ I covered, so in order to find $s(t)$ I should calculate the antiderivative of $v(t)$:

$$s(t) = \int v(t)\,dt = \int \sqrt{4+7t}\,dt = \int (4+7t)^{\frac{1}{2}}\,dt = \frac{2}{21}(4+7t)^{\frac{3}{2}} + C$$

This problem is a good example of why it is necessary to add an arbitrary constant to antiderivatives: I started my walk at $t = 0$ with zero distance covered, so $s(0) = 0$, and I can use this fact (which is an example of an initial condition) to calculate the value of C:

$$s(0) = 0 \quad \Longrightarrow \quad \frac{2}{21}(4+0)^{\frac{3}{2}} + C = \frac{2}{21} \cdot 4\sqrt{4} + C = \frac{16}{21} + C = 0 \quad \Longrightarrow \quad C = -\frac{16}{21}$$

Conclusion: $s(t)$ is given by

$$s(t) = \frac{2}{21}(4+7t)^{\frac{3}{2}} - \frac{16}{21}$$

and the distance I covered in 3 hours is

$$s(3) = \frac{2}{21}(4+21)^{\frac{3}{2}} - \frac{16}{21} = \frac{2}{21} \cdot 25\sqrt{25} - \frac{16}{21} = \frac{250}{21} - \frac{16}{21} = \frac{234}{21} = \frac{78}{7} \text{ kilometres}$$

Exercise 10.9.

a) $$f(x) = \frac{3}{x^2-1} = \frac{\alpha}{x+1} + \frac{\beta}{x-1} = \frac{\alpha(x-1)+\beta(x+1)}{(x+1)(x-1)} = \frac{(\alpha+\beta)x+(\beta-\alpha)}{x^2-1} \quad \Longrightarrow \quad \begin{cases} \alpha = -\frac{3}{2} \\ \beta = \frac{3}{2} \end{cases}$$

$$\Longrightarrow \quad \int f(x)\,dx = -\frac{3}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx = -\frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C = \frac{3}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

b)

$$\begin{array}{rcccccccccl}
x-3 & / & 2x^3 & & & - & x & + & 1 & \backslash & 2x^2+6x+17+\frac{52}{x-3} \\
 & - & 2x^3 & - & 6x^2 & & & & & & \\ \hline
 & & & & 6x^2 & - & x & + & 1 & & \\
 & & & - & 6x^2 & - & 18x & & & & \\ \hline
 & & & & & & 17x & + & 1 & & \\
 & & & & & - & 17x & - & 51 & & \\ \hline
 & & & & & & & & 52 & &
\end{array}$$

$$\Longrightarrow \quad \int f(x)\,dx = \int \left(2x^2 + 6x + 17 + \frac{52}{x-3}\right) dx = \frac{2}{3}x^3 + 3x^2 + 17x + 52\ln|x-3| + C$$

Exercise 11.1.

$$\int_{e^2}^{e^3} \frac{5}{2x}\,dx = \left[\frac{5}{2}\ln|x|\right]_{e^2}^{e^3} = \frac{5}{2}\ln\left(e^3\right) - \frac{5}{2}\ln\left(e^2\right) = \frac{5}{2}\cdot 3 - \frac{5}{2}\cdot 2 = \frac{5}{2}$$

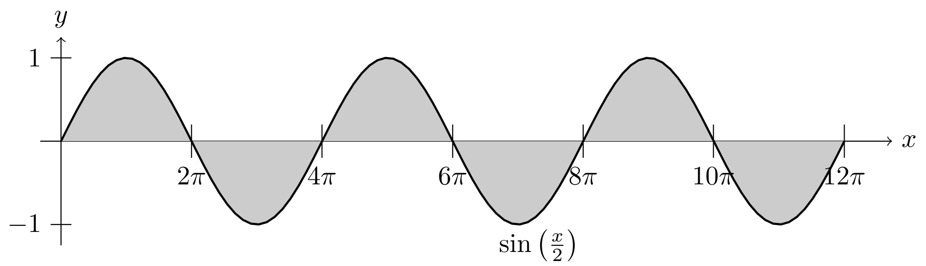
This integral represents the area between the graph of $f(x) = \frac{5}{2x}$ and the x-axis from $x = e^2$ to $x = e^3$.

Exercise 11.2.

$$\text{Area}_{(3\text{ periods})} = 3 \cdot \text{Area}_{(1\text{ period})} = 6 \cdot \text{Area}_{(\text{half period})}$$

$$A_{(\text{half period})} = \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos\left(\frac{x}{2}\right)\right]_0^{2\pi} = -2\cos(\pi) + 2\cos(0) = -2 \cdot -1 + 2 \cdot 1 = 2 + 2 = 4$$

$$\text{Total area of 3 periods} = 6 \cdot 4 = 24$$



Exercise 11.3.

$$\text{Area} = \int_0^1 \left(e^{4x} - 2\sqrt{x}\right) dx = \left[\frac{1}{4}e^{4x} - \frac{4}{3}x\sqrt{x}\right]_0^1 = \frac{1}{4} \cdot e^4 - \frac{4}{3} \cdot 1 \cdot \sqrt{1} - \frac{1}{4} \cdot e^{4 \cdot 0} + \frac{4}{3} \cdot 0 \cdot \sqrt{0}$$

$$= \frac{e^4}{4} - \frac{4}{3} - \frac{1}{4} = \frac{e^4}{4} - \frac{19}{12} = \frac{3e^4 - 19}{12}$$

Exercise 11.4. The graphs of $y = x$ and $y = x^2$ intersect right in the middle of the region $0 \leq x \leq 2$, namely at $x = 1$, so we will have to split our integral into a left part and a right part and carefully check which of the two graphs is the upper and which is the lower:

$$A_{\text{left}} = \int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

$$A_{\text{right}} = \int_1^2 \left(x^2 - x\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \frac{8}{3} - 2 - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{5}{6}$$

$$A_{\text{total}} = 1$$

Exercise 11.5. The function to be integrated is the composition of the functions $y = \sqrt{u}$ and $u = 1 - x^2$.

- David mistakenly tries to apply the reverse chain rule, because he does not realise that you can only use the reverse chain rule if u is a linear function of x. In addition, David made a mistake in calculating the result, because the answer to his integral is $-\frac{1}{3}$.
- Cheers to Frank for realising that $1 - x^2 = (1 - x)(1 + x)$, but he invented his own 'product rule for antidifferentiation' ($\int f(x) \cdot g(x)\, dx = \int f(x)\, dx \cdot \int g(x)\, dx$), which is not correct at all. In fact, as you hopefully remember from the previous chapter, there exists no product rule for antidifferentiation.
- Sammy realises that the graph of $f(x) = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ is a quarter-circle with radius 1, and the area of a quarter-circle with radius 1 is $\frac{\pi}{4}$. Well done Sammy!

Exercise 11.6.

$$\int_0^{18} \frac{15}{2} \sin\left(\frac{5}{28}t\right) dt = \frac{15}{2}\left[-\frac{28}{5}\cos\left(\frac{5}{28}t\right)\right]_0^{18} = -42\left(\cos\left(\frac{45}{14}\right) - 1\right) = 42\left(1 - \cos\left(\frac{45}{14}\right)\right)$$

This integral cannot be evaluated further, but it is approximately equal to 83.89 metres. If you look at the original (v, t)-diagram, you can see that the area under the graph of $v(t)$ is indeed a bit less than half of the whole diagram of 10 m/s $\cdot$ 20 s = 200 m.

By the way, the reason why you can use integration to calculate the distance from $v(t)$ is because you can calculate the very small displacement ds in a tiny time interval dt by multiplying dt with $v(t)$, which you can assume to be constant because dt is so tiny. Now, $ds = v(t) \cdot dt$ is the area of a very narrow rectangle under the graph of $v(t)$ in the (v, t)-diagram, and the principle behind integration is that you are essentially adding up all these narrow rectangles to find the total area under the graph of $v(t)$.

Note that you can also solve this problem using antidifferentiation, just like in exercise 8 from the previous chapter: your answer should be the same as long as you make sure that you use the same initial condition: $s(t = 0) = 0$.

Exercise 11.7.

$$V = \pi \cdot \int_1^{17} \left(\frac{6}{\sqrt{x}}\right)^2 dx = \pi \cdot \int_1^{17} \frac{36}{x}\, dx = \pi \cdot \Big[36 \ln|x|\Big]_1^{17} = 36\pi \ln(17)$$

Exercise 12.1.

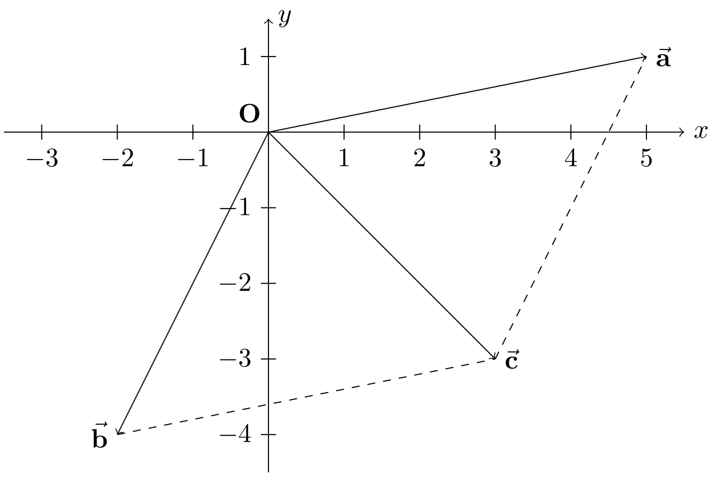
a) $\vec{\mathbf{c}} = \vec{\mathbf{a}} + 9\vec{\mathbf{b}} = (8,3) + 9(-3,6) = (8,3) + (-27,54) = (-19,57)$

b) $\vec{\mathbf{p}} = -6\vec{\mathbf{a}} + 4\vec{\mathbf{b}} = -6(8,3) + 4(-3,6) = (-48,-18) + (-12,24) = (-60,6)$

c) $\vec{\mathbf{q}} = 7\vec{\mathbf{b}} - 2\vec{\mathbf{a}} = 7(-3,6) - 2(8,3) = (-21,42) - (16,6) = (-37,36)$

d) $\vec{\mathbf{r}} = -10\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = -10(8,3) + 3(-3,6) = (-80,-30) + (-9,18) = (-89,-12)$
so $|\vec{\mathbf{r}}| = \sqrt{(-89)^2 + (-12)^2} = \sqrt{8065}$.

Exercise 12.2.



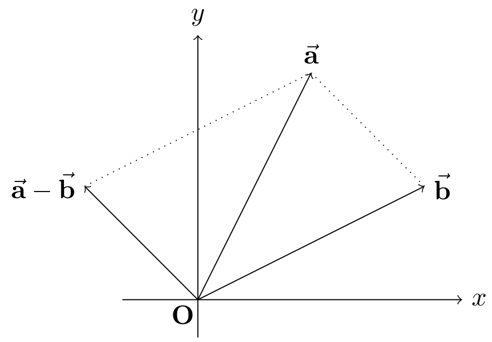
a) + c)

b) $\cos\varphi = \dfrac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|} = \dfrac{(5,1) \bullet (-2,-4)}{|(5,1)| \cdot |(-2,-4)|} = \dfrac{-14}{\sqrt{26} \cdot \sqrt{20}} = -\dfrac{7\sqrt{130}}{130}$
so φ is approximately 128 degrees (using the $\cos^{-1}$ button on my calculator).

c) $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} = (5,1) + (-2,-4) = (5-2, 1-4) = (3,-3)$

d) $|\vec{\mathbf{a}}| = |(5,1)| = \sqrt{5^2 + 1^2} = \sqrt{26}$
$|\vec{\mathbf{b}}| = |(-2,-4)| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$
$|\vec{\mathbf{c}}| = |(3,-3)| = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$.
The distance between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given by $|\vec{\mathbf{a}} - \vec{\mathbf{b}}| = |(5-(-2), 1-(-4))| = |(7,5)| = \sqrt{7^2 + 5^2} = \sqrt{74}$

Exercise 12.3. If you add $\vec{\mathbf{b}}$ to the desired vector, the result must equal $\vec{\mathbf{a}}$, so you should draw a parallelogram:



Exercise 12.4.

a) $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = (4,2) \bullet (2,3) = 4 \cdot 2 + 2 \cdot 3 = 8 + 6 = 14$

b) $|\vec{\mathbf{a}}| = |(4,2)| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$|\vec{\mathbf{b}}| = |(2,3)| = \sqrt{2^2 + 3^2} = \sqrt{13}$

c) The distance from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$ is $|\vec{\mathbf{a}} - \vec{\mathbf{b}}| = |(2,-1)| = \sqrt{5}$.

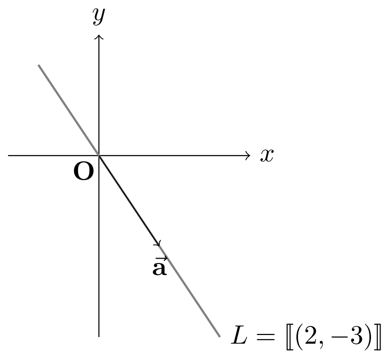
d) $\cos\varphi = \dfrac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|} = \dfrac{14}{2\sqrt{5} \cdot \sqrt{13}} = \dfrac{7}{\sqrt{65}}$

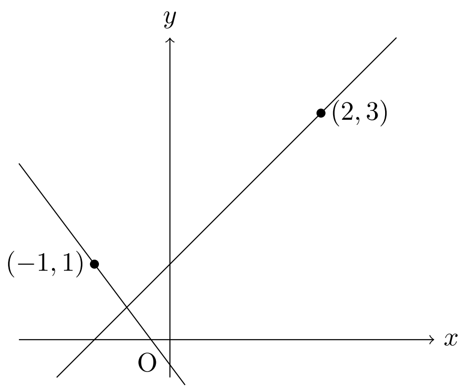
e) The projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is $\dfrac{\vec{\mathbf{b}} \bullet \vec{\mathbf{a}}}{\vec{\mathbf{a}} \bullet \vec{\mathbf{a}}} \cdot \vec{\mathbf{a}} = \dfrac{(2,3) \bullet (4,2)}{(4,2) \bullet (4,2)} \cdot (4,2) = \dfrac{2 \cdot 4 + 3 \cdot 2}{4 \cdot 4 + 2 \cdot 2} \cdot (4,2) = \dfrac{14}{20}(4,2) = \left(\dfrac{14}{5}, \dfrac{7}{5}\right)$

Exercise 12.5. The vector $\vec{\mathbf{a}}$ can be written as

$$\vec{\mathbf{a}} = 2\vec{\mathbf{e_x}} - 3\vec{\mathbf{e_y}} = 2(1,0) - 3(0,1) = (2,-3)$$

The line $[\![\vec{\mathbf{a}}]\!]$ is the collection of all vectors $t \cdot \vec{\mathbf{a}} = t \cdot (2,-3)$ with t any real number. You can also write this line as $L = [\![(2,-3)]\!]$. Since $\vec{\mathbf{a}}$ is the direction vector of L, you can construct L by drawing $\vec{\mathbf{a}}$ and then extending it infinitely long in two directions: in the direction into which $\vec{\mathbf{a}}$ points and in the opposite direction.





Exercise 12.6. The intersection P lies on both lines, so

$$\begin{cases} P = (-1,1) + t_1 \cdot (3,-4) & = (-1 + 3t_1, 1 - 4t_1) \\ P = (2,3) + t_2 \cdot (1,1) & = (2 + t_2, 3 + t_2) \end{cases}$$

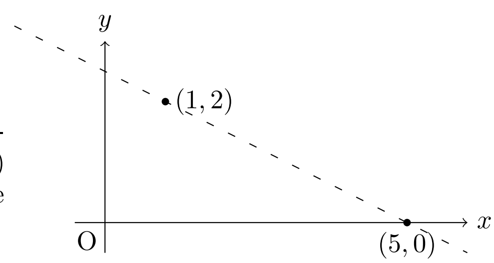
A little bit of algebra:

$$\left.\begin{array}{lll} -1 + 3t_1 = 2 + t_2 & \Longrightarrow & t_2 = -3 + 3t_1 \\ 1 - 4t_1 = 3 + t_2 & \Longrightarrow & t_2 = -2 - 4t_1 \end{array}\right\} \Longrightarrow$$

$$-3 + 3t_1 = -2 - 4t_1 \implies t_1 = \tfrac{1}{7} \implies P = (-\tfrac{4}{7}, \tfrac{3}{7})$$

Exercise 12.7. A point on this line can be found by adding a multiple of the direction vector $(5,0) - (1,2)$ to $(1,2)$:

$$(1,2) + t \cdot ((5,0) - (1,2)) = \boxed{(1,2) + t \cdot (4,-2)}$$



Remark. This line has infinitely many different parametric representations, because you can choose any vector parallel to $(4,-2)$ to be the direction vector, and any vector on the line to be the supporting vector. For example:

$$\boxed{(1,2) + t(2,-1)} \qquad \text{or} \qquad \boxed{(3,1) + t(-2,1)}$$

Exercise 12.8. Every point (x, y) on the line can be written as $(x, y) = (1, 3) + t \cdot (5, 2) = (1 + 5t, 3 + 2t)$. Let's eliminate t:

$$\left.\begin{array}{lll} x = 1 + 5t & \Longrightarrow & 2x - 2 = 10t \\ y = 3 + 2t & \Longrightarrow & 5y - 15 = 10t \end{array}\right\} \Longrightarrow 2x - 2 = 5y - 15 \Longrightarrow y = \frac{2}{5}x + \frac{13}{5}$$

Exercise 12.9. $y = 7 - 3x \Longrightarrow (x, y) = (x, 7 - 3x) = (0, 7) + x \cdot (1, -3)$

This way we've found a parametric representation of the line: $(0, 7) + t \cdot (1, -3)$.

Exercise 12.10.

a) The pussycat runs $|(6, 10) - (2, 7)| = |(4, 3)| = \sqrt{4^2 + 3^2} = 5$ metres in 1 second, so its speed is 5 m/sec.

b) After t sec the pussycat is located at

$$\text{starting point} + t \cdot (\text{end point} - \text{starting point}) = (2, 7) + t \cdot (4, 3) = (2 + 4t, 7 + 3t)$$

c) Its velocity is $(4, 3)$ metres per second because each second it runs 4 metres in x-direction and 3 metres in y-direction. It's no coincidence that the derivative of $(2 + 4t, 7 + 3t)$ with respect to t is precisely $(4, 3)$.

Exercise 12.11. The intersection can both be written as $t_1 \cdot (3, 2)$ and as $(1, 3) + t_2 \cdot (1, -1)$, so:

$$t_1(3, 2) = (1, 3) + t_2(1, -1) \quad \Longrightarrow \quad (3t_1, 2t_1) = (1 + t_2, 3 - t_2) \quad \Longrightarrow \quad \begin{cases} 3t_1 = 1 + t_2 \\ 2t_1 = 3 - t_2 \end{cases}$$

This system of equations leads to $t_1 = \frac{4}{5}$ and $t_2 = \frac{7}{5}$, so the desired point is $(\frac{12}{5}, \frac{8}{5})$.

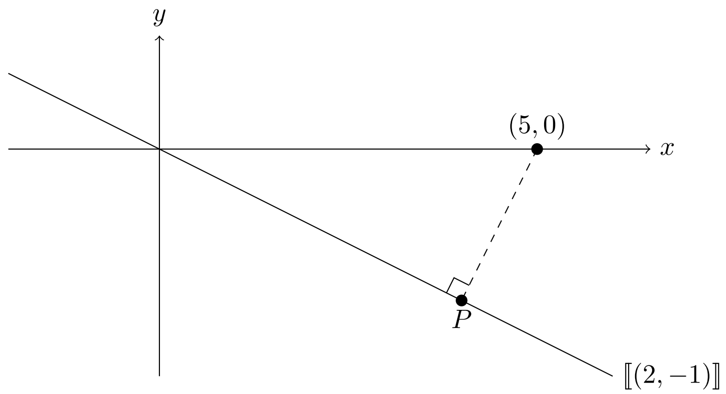
Exercise 12.12. $\cos\alpha = \dfrac{(3, 1) \bullet (2, -1)}{|(3, 1)| \cdot |(2, -1)|} = \dfrac{5}{\sqrt{10} \cdot \sqrt{5}} = \dfrac{1}{\sqrt{2}}$ implies that $\boxed{\alpha = 45^\circ}$.

Exercise 12.13. If $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$, then $\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = 0$. Substituting this into the projection formula yields

$$\vec{\mathbf{p}} = \frac{\vec{\mathbf{a}} \bullet \vec{\mathbf{b}}}{\vec{\mathbf{b}} \bullet \vec{\mathbf{b}}} \cdot \vec{\mathbf{b}} = \frac{0}{\vec{\mathbf{b}} \bullet \vec{\mathbf{b}}} \cdot \vec{\mathbf{b}} = 0 \cdot \vec{\mathbf{b}}$$

and $0 \cdot \vec{\mathbf{b}} = (0, 0)$ for any vector $\vec{\mathbf{b}}$ because of the rules for scalar multiplication.

Exercise 12.14. Let's draw the situation first:



The point P we are looking for is exactly where the projection of $(5, 0)$ onto the line is pointing to! In order to calculate this projection, we can just project $(5, 0)$ onto any vector along the line, for example $(2, -1)$:

$$\vec{\mathbf{p}} = \frac{(5, 0) \bullet (2, -1)}{(2, -1) \bullet (2, -1)} \cdot (2, -1) = \frac{10}{5} \cdot (2, -1) = (4, -2)$$

so the closest point on $[\![(2, -1)]\!]$ to $(5, 0)$ is $(4, -2)$.

CHAPTER 14

Test yourself

Maximum time per multiple choice test: 30 minutes. No calculators allowed; just use your pen, scrap paper and everything you have learned so far.

Test 1

1. The number $(-1)^{-5}$ is equal to

   a. 1 b. -1 c. 5 d. -5

2. You can simplify the fraction $\dfrac{x}{\frac{1}{3x}-\frac{1}{4x}}$ to

   a. $\dfrac{12}{x}$ b. 12 c. $12x$ d. $12x^2$

3. When does $\sqrt{x^2+9}$ equal $|x|+3$?

   a. never b. always c. sometimes d. often

4. The number $\ln 2 + \ln 3$ equals

   a. $\ln 5$ b. $\ln 6$ c. $\ln 8$ d. $\ln 9$

5. Which of the points $(3,5)$ and $(4,4)$ lie(s) within the circle $x^2+y^2=35$?

   a. only $(3,5)$ b. only $(4,4)$ c. both d. neither

6. How many sides does a quadrilateral (e.g. a square) have?

   a. $\sqrt{2}\cdot\sqrt[3]{4}\cdot\sqrt[6]{32}$ b. $\sqrt{5}+\sqrt{11}$ c. $2^{\frac{3}{2}}+2^{\frac{1}{2}}$ d. $8^{-\frac{2}{3}}$

7. The quotient $\dfrac{\sin 6}{\sin 3}$ is equal to

   a. $\sin 2$ b. $\sin 3$ c. $\cos 2$ d. $2\cos 3$

8. Which is greater, e^2 or $9^{\frac{1}{\ln 3}}$?

   a. e^2 b. $9^{\frac{1}{\ln 3}}$ c. they are equal d. it depends

9. If $x=\dfrac{1+3y}{1+y}$, then $y=$

   a. $\dfrac{3-x}{1-x}$ b. $\dfrac{1-x}{x-3}$ c. $\dfrac{3-x}{x-1}$ d. $\dfrac{1-x}{3-x}$

10. The equation of the tangent line to the graph of $y=\sqrt{4x-3}$ at $x=1$ is

    a. $y=x$ b. $y=2x-1$ c. $y=3x-2$ d. $y=4x-3$

11. The area enclosed by the x-axis, the y-axis, the graph of $y=\dfrac{1}{1+x}$ and the line $x=10$ is

    a. $\ln 8$ b. $\ln 9$ c. $\ln 10$ d. $\ln 11$

12. Which of the following equations does the line $(3,-1)+[\![(2,1)]\!]$ satisfy?

    a. $x=5+2y$ b. $x=5-2y$ c. $y=5+2x$ d. $y=5-2x$

Solutions to test 1

1b, 2d, 3c, 4b, 5c, 6a, 7d, 8c, 9b, 10b, 11d, 12a

1. x^{-5} means $\dfrac{1}{x^5}$, and x^5 means $x \cdot x \cdot x \cdot x \cdot x$, so

$$(-1)^{-5} = \frac{1}{(-1)^5} = \frac{1}{(-1)\cdot(-1)\cdot(-1)\cdot(-1)\cdot(-1)} = \frac{1}{-1} = -1$$

2. Simplify the denominator first:

$$\frac{x}{\frac{1}{3x}-\frac{1}{4x}} = \frac{x}{\frac{4}{12x}-\frac{3}{12x}} = \frac{x}{\frac{1}{12x}} = x \cdot 12x = 12x^2$$

3. This is only true for $x = 0$:

$$\sqrt{x^2+9} = |x|+3 \iff x^2+9 = (|x|+3)^2 \iff x^2+9 = x^2+6|x|+9 \iff 6|x| = 0 \iff x = 0$$

4. According to the calculation rule $\ln p + \ln q = \ln pq$ we find that

$$\ln 2 + \ln 3 = \ln(2 \cdot 3) = \ln 6$$

5. This circle has centre $(0, 0)$ and radius $\sqrt{35}$. Since

$$3^2 + 5^2 < 35 \quad \text{and} \quad 4^2 + 4^2 < 35,$$

the distance from the origin to each of these points is smaller than $\sqrt{35}$, so both points lie within the circle.

6. A quadrilateral has four sides, and

$$\sqrt{2} \cdot \sqrt[3]{4} \cdot \sqrt[6]{32} = 2^{\frac{1}{2}} \cdot 2^{\frac{2}{3}} \cdot 2^{\frac{5}{6}} = 2^{\frac{1}{2}+\frac{2}{3}+\frac{5}{6}} = 2^{\frac{3}{6}+\frac{4}{6}+\frac{5}{6}} = 2^{\frac{12}{6}} = 2^2 = 4$$

7. We can cleverly use the formula $\sin 2x = 2 \sin x \cos x$:

$$\frac{\sin 6}{\sin 3} = \frac{2 \sin 3 \cos 3}{\sin 3} = 2 \cos 3$$

8. I use the popular formulae $p^q = e^{q \ln p}$ and $\ln x^a = a \ln x$:

$$9^{\frac{1}{\ln 3}} = e^{\frac{1}{\ln 3}\cdot \ln 9} = e^{\frac{1}{\ln 3}\cdot 2\ln 3} = e^2$$

9. $x = \dfrac{1+3y}{1+y}$ implies that $x(1+y) = 1+3y$, so

$$x + xy = 1 + 3y \implies xy - 3y = 1 - x \implies y(x-3) = 1 - x \implies y = \frac{1-x}{x-3}$$

10. The slope of $y = \sqrt{4x-3}$ at $x = 1$ is $\left[\dfrac{d}{dx}\sqrt{4x-3}\right]_{x=1} = \left[\dfrac{2}{\sqrt{4x-3}}\right]_{x=1} = 2$, so

$$y = 2x + b \xrightarrow{\text{substitute } (1,1)} b = -1 \implies y = 2x - 1$$

11. This area is equal to

$$\int_0^{10} \frac{1}{1+x}\,dx = \Big[\ln(1+x)\Big]_0^{10} = \ln 11 - \ln 1 = \ln 11$$

12. A point (x, y) on this line can be written as $(3, -1) + t \cdot (2, 1) = (3, -1) + (2t, t) = (3 + 2t, -1 + t)$, so

$$\begin{cases} x = 2t + 3 \\ y = t - 1 \end{cases} \implies \begin{cases} x = 2t + 3 \\ 2y = 2t - 2 \end{cases} \xrightarrow{\text{subtract}} x - 2y = 5 \implies x = 5 + 2y$$

Test 2

1. You can rewrite $\dfrac{1}{\sqrt{7}-\sqrt{6}}$ as

   a. $\dfrac{1}{\sqrt{7}} - \dfrac{1}{\sqrt{6}}$ b. $\dfrac{1}{\sqrt{6}} - \dfrac{1}{\sqrt{7}}$ c. $\sqrt{6}-\sqrt{7}$ d. $\sqrt{6}+\sqrt{7}$

2. When is the inequality $|x| > x$ correct?

   a. always b. never c. for positive x d. for negative x

3. Which number x satisfies $8^{-x} = 2^{x+1}$?

   a. $\dfrac{1}{4}$ b. $\dfrac{1}{2}$ c. $-\dfrac{1}{4}$ d. $-\dfrac{1}{2}$

4. The expression $\dfrac{x^2+3}{x^{-2}} - \dfrac{1}{x^{-4}}$ can be simplified to

   a. 3 b. $3x^2$ c. $\dfrac{x^2+2}{x^2}$ d. x

5. If $\sin x = \dfrac{1}{\sqrt{5}}$ then $\tan x$ equals

   a. $\dfrac{1}{2}$ b. 2 c. $\dfrac{1}{2}$ or $-\dfrac{1}{2}$ d. 2 or -2

6. The expression $\dfrac{1}{5+t} - \dfrac{1}{5-t}$ is equal to

   a. $\dfrac{2t}{t^2-25}$ b. $\dfrac{2t}{25-t^2}$ c. $\dfrac{1}{2t}$ d. $\dfrac{1}{10}$

7. I started running at 15 km/h. Exactly midway, I switched back to 10 km/h. My average speed was

   a. 11 km/h b. 12 km/h c. 13 km/h d. 14 km/h

8. The quotient $\dfrac{\sin 20}{\cos 10}$ is equal to

   a. $\tan 2$ b. $\tan 20$ c. $2\sin 10$ d. $2\cos 10$

9. $\sqrt{8}-\sqrt{2}$ can be simplified to

   a. $\sqrt{6}$ b. $\sqrt{2}$ c. $\sqrt{4}$ d. $\sqrt{3}$

10. The number $\left(\sin^2 2 + \tan^2 2 + \cos^2 2\right)^{-\frac{1}{2}}$ is equal to

    a. $\sin 2$ b. $\cos 2$ c. $\tan 2$ d. $-\cos 2$

11. The angle between the vectors $(-\sqrt{3}, 3)$ and $(\sqrt{3}, -1)$ is equal to

    a. $\dfrac{\pi}{2}$ rad b. $\dfrac{2\pi}{3}$ rad c. $\dfrac{3\pi}{4}$ rad d. $\dfrac{5\pi}{6}$ rad

12. The maximum value of e^{-3x^2} is

    a. 0 b. 1 c. e d. e^2

Solutions to test 2

1d, 2d, 3c, 4b, 5c, 6a, 7b, 8c, 9b, 10d, 11d, 12b

1. Use the square root trick from chapter 4 and multiply numerator and denominator by $\sqrt{7}+\sqrt{6}$:

$$\frac{1}{\sqrt{7}-\sqrt{6}} = \frac{\sqrt{7}+\sqrt{6}}{(\sqrt{7}-\sqrt{6})\left(\sqrt{7}+\sqrt{6}\right)} = \frac{\sqrt{7}+\sqrt{6}}{\left(\sqrt{7}\right)^2-\left(\sqrt{6}\right)^2} = \frac{\sqrt{7}+\sqrt{6}}{1}$$

2. If $x \geq 0$, the absolute value of x is equal to x, but for negative values of x its absolute value is positive, so for $x < 0$ we have $|x| > x$.

3. Use $8 = 2^3$ to rewrite 8^{-x} as 2^{-3x} (but the ln trick from chapter 6 will also work):

$$8^{-x} = 2^{x+1} \quad\Longrightarrow\quad 2^{-3x} = 2^{x+1} \quad\Longrightarrow\quad -3x = x+1 \quad\Longrightarrow\quad -4x = 1 \quad\Longrightarrow\quad x = -\frac{1}{4}$$

4. Dividing by x^{-2} is the same as multiplying by x^2, so

$$\frac{x^2+3}{x^{-2}} - \frac{1}{x^{-4}} = x^2\left(x^2+3\right) - x^4 = x^4+3x^2-x^4 = 3x^2$$

5. $\sin x = \dfrac{1}{\sqrt{5}}$ implies that

$$\cos^2 x = 1-\sin^2 x = 1-\frac{1}{5} = \frac{4}{5} \quad\Longrightarrow\quad \cos x = \pm\frac{2}{\sqrt{5}} \quad\Longrightarrow\quad \tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{\sqrt{5}}}{\pm\frac{2}{\sqrt{5}}} = \pm\frac{1}{2}$$

6. I rearrange this mess by finding a common denominator for both terms:

$$\frac{1}{5+t} - \frac{1}{5-t} = \frac{5-t}{(5+t)(5-t)} - \frac{5+t}{(5+t)(5-t)} = \frac{(5-t)-(5+t)}{(5+t)(5-t)} = \frac{-2t}{25-t^2} = \frac{2t}{t^2-25}$$

7. The first half (let's say A km) lasted $\frac{A}{15}$ hours and the second half lasted $\frac{A}{10}$ hours, so

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{2A}{\frac{A}{15}+\frac{A}{10}} = \frac{2}{\frac{1}{15}+\frac{1}{10}} = \frac{2}{\frac{2}{30}+\frac{3}{30}} = \frac{2}{\frac{5}{30}} = \frac{2}{\frac{1}{6}} = 12 \text{ km/h}$$

8. Hopefully, the formula $\sin 2x = 2\sin x\cos x$ is now part of your toolbox:

$$\frac{\sin 20}{\cos 10} = \frac{2\sin 10\cos 10}{\cos 10} = 2\sin 10$$

9. Study chapter 4 carefully if you find this difficult:

$$\sqrt{8}-\sqrt{2} = \sqrt{4\cdot 2}-\sqrt{2} = \sqrt{4}\cdot\sqrt{2}-\sqrt{2} = 2\sqrt{2}-\sqrt{2} = \sqrt{2}$$

10. Using $\sin^2 x + \cos^2 x = 1$ and $1+\tan^2 x = \frac{1}{\cos^2 x}$ (chapter 7 exercise 10), we have

$$\left(\sin^2 2+\tan^2 2+\cos^2 2\right)^{-\frac{1}{2}} = \left(1+\tan^2 2\right)^{-\frac{1}{2}} = \left(\frac{1}{\cos^2 2}\right)^{-\frac{1}{2}} = \sqrt{\cos^2 2} = |\cos 2| = -\cos 2$$

11. If we call the desired angle α, then

$$\cos\alpha = \frac{(-\sqrt{3},3)\bullet(\sqrt{3},-1)}{|(-\sqrt{3},3)|\cdot|(\sqrt{3},-1)|} = \frac{-\sqrt{3}\cdot\sqrt{3}+3\cdot -1}{\sqrt{(-\sqrt{3})^2+3^2}\cdot\sqrt{(\sqrt{3})^2+(-1)^2}} = \frac{-6}{\sqrt{12}\cdot 2} = -\frac{\sqrt{3}}{2} \quad\Longrightarrow\quad \alpha = \frac{5\pi}{6}$$

12. The derivative of $\exp(-3x^2)$ is zero if

$$\frac{d}{dx}\,e^{-3x^2} = 0 \quad\Longleftrightarrow\quad -6x\cdot e^{-3x^2} = 0 \quad\Longleftrightarrow\quad x = 0$$

so the desired maximum value is $\left[e^{-3x^2}\right]_{x=0} = e^0 = 1.$

Test 3

1. You can simplify $\dfrac{a+b}{\frac{1}{a}+\frac{1}{b}}$ to

   a. a^2+b^2 b. $\dfrac{1}{a^2+b^2}$ c. ab d. $\dfrac{1}{ab}$

2. If $x = \dfrac{3}{4}$ and $y = \dfrac{6}{7}$, then $\dfrac{x}{y}$ is equal to

   a. $\dfrac{5}{6}$ b. $\dfrac{7}{8}$ c. $\dfrac{9}{10}$ d. $\dfrac{11}{12}$

3. The maximum value of $x - x^2$ is

   a. $\dfrac{1}{2}$ b. $\dfrac{1}{3}$ c. $\dfrac{1}{4}$ d. $\dfrac{1}{5}$

4. If $\cos\alpha = \dfrac{5}{13}$ then the absolute value of $\tan\alpha$ is equal to

   a. 4.25 b. 0.375 c. 2.4 d. 1.75

5. How many real numbers x satisfy $|x-3| = 5$?

   a. none b. one c. two d. three

6. The radius of the circle $x^2 + y^2 = x$ is

   a. $\sqrt{x}$ b. 1 c. $\dfrac{1}{2}$ d. $\dfrac{1}{4}$

7. The number $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{2}+\frac{1}{3}}$ is equal to

   a. 0.3 b. 0.5 c. 0.7 d. 0.9

8. The quotient $\dfrac{\ln 8}{\ln 2}$ can be simplified to

   a. $\ln 4$ b. $\ln 6$ c. 4 d. 3

9. If you want to simplify the product $e^{2x} \cdot e^{3x} \cdot e^{4x}$, then you write:

   a. e^{9x} b. e^{24x} c. e^{9x^3} d. e^{24x^3}

10. The lines $(2,1) + [\![(-1,2)]\!]$ and $(1,3) + [\![(2,-1)]\!]$ intersect at

    a. $(2,1)$ b. $(-1,2)$ c. $(1,3)$ d. $(2,-1)$

11. The derivative of $\dfrac{1}{\cos x}$ is

    a. $\dfrac{\sin x}{\tan x}$ b. $\dfrac{\cos x}{\tan x}$ c. $\dfrac{\tan x}{\sin x}$ d. $\dfrac{\tan x}{\cos x}$

12. If my velocity at time t is given by $v(t) = \dfrac{6}{\sqrt{t+1}}$, the distance I covered from $t = 0$ is given by $s(t) =$

    a. $6\sqrt{t+1} - 6$ b. $12\sqrt{t+1} - 12$ c. $\dfrac{3}{\sqrt{t+1}} - 3$ d. $3 - \dfrac{3}{(t+1)^{\frac{3}{2}}}$

Solutions to test 3

1c, 2b, 3c, 4c, 5c, 6c, 7c, 8d, 9a, 10c, 11d, 12b

1. If you first add the fractions in the denominator, then the rest will be easy:

$$\frac{a+b}{\frac{1}{a}+\frac{1}{b}} = \frac{a+b}{\frac{b}{ab}+\frac{a}{ab}} = \frac{a+b}{\frac{b+a}{ab}} = \frac{1}{\frac{1}{ab}} = ab$$

2. Dividing by $\dfrac{6}{7}$ is the same as multiplying by $\dfrac{7}{6}$:

$$\frac{x}{y} = \frac{3/4}{6/7} = \frac{3}{4}\cdot\frac{7}{6} = \frac{3\cdot 7}{4\cdot 6} = \frac{7}{4\cdot 2} = \frac{7}{8}$$

3. You could of course find the maximum of the function $f(x) = x - x^2$ using differentiation, but completing the square is much more elegant:

$$x - x^2 = -(x^2 - x) = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$$

Since squares are always positive, the *minimum* value of $(x-\frac{1}{2})^2$ is zero, so the *maximum* value of $-(x-\frac{1}{2})^2$ is zero and you can easily see what the correct answer should be.

4. $\cos\alpha = \dfrac{5}{13}$ implies that

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\frac{25}{169}} = \pm\sqrt{\frac{144}{169}} = \pm\frac{12}{13} \quad\Longrightarrow\quad |\tan\alpha| = \frac{12/13}{5/13} = \frac{12}{5} = 2.4$$

5. This equation is only true for the numbers 8 and -2, because

$$|x-3| = 5 \quad\Longleftrightarrow\quad x - 3 = \pm 5 \quad\Longleftrightarrow\quad x = 3 \pm 5$$

6. Use completing the square to write the equation of this circle in standard form:

$$x^2 - x + y^2 = 0 \quad\Longrightarrow\quad x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \quad\Longrightarrow\quad \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

7. Simplify both the numerator and the denominator:

$$\frac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{2}+\frac{1}{3}} = \frac{\frac{4}{12}+\frac{3}{12}}{\frac{3}{6}+\frac{2}{6}} = \frac{\frac{7}{12}}{\frac{5}{6}} = \frac{\frac{7}{12}\cdot 12}{\frac{5}{6}\cdot 12} = \frac{7}{10}$$

8. By means of the calculation rule $\ln a^p = p\ln a$ I find

$$\frac{\ln 8}{\ln 2} = \frac{\ln 2^3}{\ln 2} = \frac{3\ln 2}{\ln 2} = 3$$

9. I use the calculation rule $a^p \cdot a^q = a^{p+q}$: $\quad e^{2x}\cdot e^{3x}\cdot e^{4x} = e^{2x+3x+4x} = e^{9x}$

10. A point on the line $(2,1) + [\![(-1,2)]\!]$ satisfies $(x,y) = (2-t_1, 1+2t_1)$ and a point on the line $(1,3) + [\![(2,-1)]\!]$ satisfies $(x,y) = (1+2t_2, 3-t_2)$. At the point of intersection their coordinates must be equal:

$$\begin{cases} 2 - t_1 = 1 + 2t_2 \\ 1 + 2t_1 = 3 - t_2 \end{cases} \Longrightarrow \begin{cases} 4 - 2t_1 = 2 + 4t_2 \\ 1 + 2t_1 = 3 - t_2 \end{cases} \overset{\text{add}}{\Longrightarrow} 5 = 5 + 3t_2 \Longrightarrow t_2 = 0 \Longrightarrow (x,y) = (1,3)$$

11. Use the quotient rule: $\displaystyle \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x\cdot\cos x} = \frac{\tan x}{\cos x}$

12. $s(t)$ is an antiderivative of $v(t)$ (see chapter 10 exercise 8), so

$$s(t) = \int v(t)\,dt = \int \frac{6}{\sqrt{t+1}}\,dt = \int 6(t+1)^{-\frac{1}{2}}\,dt = 12(t+1)^{\frac{1}{2}} + C \quad\overset{s(0)=0}{\Longrightarrow}\quad s(t) = 12\sqrt{t+1} - 12$$

Test 4

1. If $A = B^{-\frac{3}{2}}$, then $B =$

   a. $A^{\frac{3}{2}}$ b. $A^{\frac{2}{3}}$ c. $A^{-\frac{3}{2}}$ d. $A^{-\frac{2}{3}}$

2. If $x^2 + y^2 = 50$ and $x + y = 8$, then $x =$

   a. 1 b. 7 c. 1 of 7 d. 17

3. You can write $\dfrac{x - y}{\sqrt{x} - \sqrt{y}}$ as

   a. $\sqrt{x} - \sqrt{y}$ b. $\sqrt{x} + \sqrt{y}$ c. $\sqrt{y} - \sqrt{x}$ d. $\dfrac{1}{\sqrt{x} + \sqrt{y}}$

4. If $\dfrac{6}{x} = \dfrac{1}{a} + \dfrac{1}{2a}$ then x equals

   a. $12a$ b. $9a$ c. $4a$ d. $2a$

5. The number $3^{-\frac{1}{3}} \cdot 9^{\frac{2}{3}}$ is equal to

   a. 3 b. $\sqrt{3}$ c. 9 d. 1

6. Which angles α between 0 and 90 degrees satisfy $\sqrt{1 + \sin 2\alpha} = \sin\alpha + \cos\alpha$?

   a. none b. one c. two d. very many

7. How many real numbers x satisfy $|x + 1| = |x + 3|$?

   a. none b. one c. two d. very many

8. The derivative of $\dfrac{x - 1}{x}$ is

   a. 1 b. -1 c. $\dfrac{1}{x^2}$ d. $-\dfrac{1}{x^2}$

9. If $w = e^{-t \ln 4}$, then $\sqrt{w} =$

   a. 2^t b. 2^{-t} c. t^2 d. t^{-2}

10. If $A + 3B = 10$ and $3A + 2B = 23$ then $B =$

    a. 1 b. 2 c. 3 d. 4

11. You can simplify $\cos\left(t + \dfrac{\pi}{6}\right) - \cos\left(t - \dfrac{\pi}{6}\right)$ to

    a. $\cos t$ b. $\sin t$ c. $-\cos t$ d. $-\sin t$

12. Which of the following expressions is an antiderivative of $\exp(2 - x)$?

    a. e^{2-x} b. $-e^{2-x}$ c. $2e^{2-x}$ d. $-2e^{2-x}$

Solutions to test 4

1d, 2c, 3b, 4c, 5a, 6d, 7b, 8c, 9b, 10a, 11d, 12b

1. Using the calculation rule $(B^p)^q = B^{pq}$ we find

$$A = B^{-\frac{3}{2}} \quad \Longrightarrow \quad B = B^1 = B^{(-\frac{3}{2})\cdot(-\frac{2}{3})} = \left(B^{-\frac{3}{2}}\right)^{-\frac{2}{3}} = A^{-\frac{2}{3}}$$

2. You should substitute $y = 8 - x$ into $x^2 + y^2 = 50$:

$$x^2 + (8-x)^2 = 50 \quad \Longrightarrow \quad 2x^2 - 16x + 14 = 0 \quad \Longrightarrow \quad x^2 - 8x + 7 = 0 \quad \Longrightarrow \quad (x-1)(x-7) = 0$$

3. You could use the square root trick from chapter 3, but if you cleverly apply the notable product $p^2 - q^2 = (p+q)(p-q)$, you can substitute $x - y = (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$ in the numerator:

$$\frac{x-y}{\sqrt{x}-\sqrt{y}} = \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \sqrt{x}+\sqrt{y}$$

4. $\dfrac{6}{x} = \dfrac{1}{a} + \dfrac{1}{2a}$ implies that

$$\frac{6}{x} = \frac{2}{2a} + \frac{1}{2a} = \frac{3}{2a} \quad \Longrightarrow \quad \frac{x}{6} = \frac{2a}{3} \quad \Longrightarrow \quad x = 4a$$

5. Use the calculation rules $(x^y)^z = x^{yz}$ and $x^p \cdot x^q = x^{p+q}$:

$$3^{-\frac{1}{3}} \cdot 9^{\frac{2}{3}} = 3^{-\frac{1}{3}} \cdot \left(3^2\right)^{\frac{2}{3}} = 3^{-\frac{1}{3}} \cdot 3^{\frac{4}{3}} = 3^{-\frac{1}{3}+\frac{4}{3}} = 3^1 = 3$$

6. This equation holds for all angles α, because

$$(\sin\alpha + \cos\alpha)^2 = \sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = 1 + \sin 2\alpha$$

7. Only the number $x = -2$ satisfies this equation, because

$$|x+1| = |x+3| \quad \Longrightarrow \quad x+1 = \pm(x+3) \quad \Longrightarrow \quad x+1 = -(x+3) \quad \Longrightarrow \quad 2x = -4 \quad \Longrightarrow \quad x = -2$$

8. This is possible using the quotient rule, but it is much easier to use:

$$\frac{d}{dx}\frac{x-1}{x} = \frac{d}{dx}\left(1 - \frac{1}{x}\right) = \frac{1}{x^2}$$

9. I use familiar calculation rules such as $\ln(x^a) = a\ln x$ and $a^b = e^{b\ln a}$:

$$w = e^{-t\ln 4} = e^{-2t\ln 2} = 2^{-2t} \quad \Longrightarrow \quad \sqrt{w} = w^{\frac{1}{2}} = \left(2^{-2t}\right)^{\frac{1}{2}} = 2^{\left(-2t\cdot\frac{1}{2}\right)} = 2^{-t}$$

10. Multiply the first equation by 3, and then subtract the second equation:

$$\boxed{3A + 9B = 30} - \boxed{3A + 2B = 23} \quad \Longrightarrow \quad 7B = 7 \quad \Longrightarrow \quad B = 1$$

11. Using two of the angle sum and difference identities from chapter 7 I can rewrite this as

$$\left(\cos t\cos\frac{\pi}{6} - \sin t\sin\frac{\pi}{6}\right) - \left(\cos t\cos\frac{\pi}{6} + \sin t\sin\frac{\pi}{6}\right) = -2\sin t\sin\frac{\pi}{6} = -\sin t$$

12. You can verify this by differentiation:

$$\frac{d}{dx}\left(-e^{2-x}\right) = -e^{2-x}\cdot -1 = e^{2-x} \quad \Longrightarrow \quad \int e^{2-x}\,dx = -e^{2-x} + C$$

Test 5

1. The points (x, y) satisfying $\dfrac{x+y+1}{x-3y-1} = 2$ all lie on the line with equation

   a. $x - 7y = 3$ b. $7x - y = 3$ c. $x - 3y = 7$ d. $3x - y = 7$

2. The angle between the minute hand and hour hand on a clock displaying 11:50h is

   a. 54 degrees b. 55 degrees c. 56 degrees d. 57 degrees

3. For positive real numbers t the number $e^{-\frac{\ln t}{2}}$ is equal to

   a. $\sqrt{t}$ b. $-\sqrt{t}$ c. $\dfrac{1}{\sqrt{t}}$ d. $-\dfrac{1}{\sqrt{t}}$

4. If $\alpha + \beta^2 = 10$ and $3\alpha - 5\beta^2 = 6$, then $\alpha =$

   a. 1 b. 3 c. 5 d. 7

5. You can simplify $\left(\dfrac{1}{x} + \dfrac{1}{1-x}\right)^{-1}$ to

   a. $x^2 - x$ b. $x - x^2$ c. $x^2 - 1$ d. $1 - x^2$

6. Which of the following do you suspect to equal $\sin\left(A + \dfrac{\pi}{2}\right)$?

   a. $\cos A$ b. $\sin A$ c. $-\cos A$ d. $-\sin A$

7. You can condense $\dfrac{x^3 - 27}{x - 3}$ to

   a. $x^2 - 9$ b. $x^2 + 9$ c. $x^2 + 3x + 9$ d. $x^2 - 3x + 9$

8. How many real solutions x can you find for the equation $e^{2x} + e^x = 12$?

   a. zero b. one c. two d. three

9. If $A = 1 + \dfrac{1}{\tan^2 x}$ then $A^{-\frac{1}{2}} =$

   a. $\cos x$ b. $\sin x$ c. $|\cos x|$ d. $|\sin x|$

10. Which of the following numbers lies closest to $\dfrac{\ln 2.5 + \ln 0.4}{\ln 1.3 + \ln 1.7}$?

    a. 0 b. 1 c. 2 d. 3

11. The derivative of $\cos^2 x$ is equal to

    a. $-\sin^2 x$ b. $2\cos x$ c. $-2\sin x$ d. $-\sin 2x$

12. The area enclosed by the graphs of $y = 3x^2$ and $y = 6 - 3x^2$ is equal to

    a. 3 b. 4 c. 8 d. 12

Solutions to test 5

1a, 2b, 3c, 4d, 5b, 6a, 7c, 8b, 9d, 10a, 11d, 12c

1. The coordinates of these points satisfy $x + y + 1 = 2(x - 3y - 1)$, so

$$x + y + 1 = 2x - 6y - 2 \quad \Longrightarrow \quad -x + 7y + 3 = 0 \quad \Longrightarrow \quad 3 = x - 7y$$

2. The angle between two consecutive numbers is $\frac{360}{12} = 30$ degrees, so the requested angle is

$$30 + \frac{5}{6} \cdot 30 = 55 \text{ degrees}$$

3. Since $a \ln t = \ln t^a$, we have

$$e^{-\frac{\ln t}{2}} = e^{-\frac{1}{2}\ln t} = e^{\ln\left(t^{-\frac{1}{2}}\right)} = t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}}$$

4. Multiply the first equation by 5, and then add the second equation:

$$\boxed{5\alpha + 5\beta^2 = 50} + \boxed{3\alpha - 5\beta^2 = 6} \quad \Longrightarrow \quad 8\alpha = 56 \quad \Longrightarrow \quad \alpha = 7$$

5. Calculate $\dfrac{1}{x} + \dfrac{1}{1-x}$ first:

$$\left(\frac{1}{x} + \frac{1}{1-x}\right)^{-1} = \left(\frac{1-x}{x(1-x)} + \frac{x}{x(1-x)}\right)^{-1} = \left(\frac{1}{x(1-x)}\right)^{-1} = x(1-x) = x - x^2$$

6. You can deduce this from the graphs of sin and cos or using $\sin(p+q) = \sin p \cos q + \cos p \sin q$:

$$\sin\left(A + \frac{\pi}{2}\right) = \sin A \cos\frac{\pi}{2} + \cos A \sin\frac{\pi}{2} = \cos A$$

7. Long division leads to answer c, which you can verify as follows:

$$(x-3)(x^2+3x+9) = x(x^2+3x+9) - 3(x^2+3x+9) = x^3 + 3x^2 + 9x - 3x^2 - 9x - 27 = x^3 - 27$$

8. You can solve this quadratic equation in e^x using the quadratic formula or by factorisation:

$$e^{2x} + e^x - 12 = 0 \quad \Longleftrightarrow \quad (e^x - 3)(e^x + 4) = 0 \quad \Longleftrightarrow \quad e^x = 3 \quad \Longleftrightarrow \quad x = \ln 3$$

9. I simplify the expression for A first:

$$A = 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \Longrightarrow \quad A^{-\frac{1}{2}} = \frac{1}{\sqrt{A}} = \sqrt{\sin^2 x} = |\sin x|$$

10. According to the calculation rule $\ln p + \ln q = \ln pq$ the numerator is

$$\ln 2.5 + \ln 0.4 = \ln(2.5 \cdot 0.4) = \ln 1 = 0$$

11. Use the chain rule to differentiate $(\cos x)^2$ and then use $\sin 2x = 2 \sin x \cos x$ to simplify:

$$\frac{d}{dx}(\cos x)^2 = 2(\cos x) \cdot (-\sin x) = -2\sin x \cos x = -\sin 2x$$

12. These graphs intersect where $x = \pm 1$, and on the domain $-1 \leq x \leq 1$ we have $6 - 3x^2 \geq 3x^2$, so the desired area is

$$\int_{-1}^{1} (6 - 3x^2 - 3x^2)\, dx = \int_{-1}^{1} (6 - 6x^2)\, dx = \Big[6x - 2x^3\Big]_{-1}^{1} = 6 - 2 - (-6 + 2) = 8$$